PSEUDOHOLOMORPHIC CURVES AND THE SYMPLECTIC ISOTOPY PROBLEM

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Abstract. The deformation problem for pseudoholomorphic curves and related geometrical properties of the total moduli space of pseudoholomorphic curves are studied. A sufficient condition for the saddle point property of the total moduli space is established. The local symplectic isotopy problem is formulated and solved for the case of imbedded pseudoholomorphic curves. It is shown that any two symplectically imbedded surfaces $\Sigma_0, \Sigma_1 \subset \mathbb{CP}^2$ of the same degree $d \leqslant 6$ are symplectically isotopic.

0. Introduction

The symplectic isotopy problem can be formulated as follows:

For a given symplectic 4-dimensional manifold (X,ω) *and symplectically imbedded compact connected oriented surfaces* $\Sigma_0, \Sigma_1 \subset X$ *in the same homology class* $[A] \in \mathsf{H}_2(X,\mathbb{Z})$, *does there exist an isotopy* $\{\Sigma_t\}_{t\in[0,1]}$ *connecting* Σ_0 *with* Σ_1 *such that all* Σ_t *are also symplectically imbedded?*

In this case Σ_0 and Σ_1 are called *symplectically isotopic*. Note that by the *genus formula* for pseudoholomorphic curves (see *Section 1*) Σ_0 and Σ_1 have the same genus. The example of Fintushel and Stern [Fi-St] (see *Section 6* for details) shows that in general the answer is negative. On the other hand, Sikorav [Sk-3] gives an affirmative answer in the case of surfaces of positive degree $d \leqslant 3$ in $\mathbb{CP}^2$. So it is natural to ask under which conditions on (X,ω) and $[A] \in \mathsf{H}_2(X,\mathbb{Z})$ the symplectic isotopy does exist.

In this paper techniques involving pseudoholomorphic curves are developed in directions needed for solving the symplectic isotopy problem. The main result is the following

Theorem 1. *Two symplectically imbedded compact connected oriented surfaces* $\Sigma_0, \Sigma_1 \subset \mathbb{CP}^2$ *of the same positive degree* $d \leqslant 6$ *are symplectically isotopic.*

The proof is based on the solution of two problems which are closely related to the symplectic isotopy problem and concern the geometry of the total moduli space of pseudoholomorphic curves. The first result is a sufficient condition for the *saddle point property* of the total moduli space of pseudoholomorphic curves. This removes one of the obstacles to the existence of a symplectic isotopy and is applied to solve the *local symplectic isotopy problem* for imbedded pseudoholomorphic curves. The latter appears as a necessary part of the global problem.

The obtained progress gives hope that the symplectic isotopy problem has an affirmative solution in the case $c_1(X,\omega)[A] > 0$. Note that compact symplectic 4-manifolds with this property are classified: Up to the case when $[A]$ is represented by an exceptional sphere, these are symplectic blow-ups of a rational or ruled symplectic manifold, see [McD-Sa-3], *Corollary 1.5*, and also *Section 6*.

0.1. Overview of main results.

0.1. Overview of main results. There is essentially only one known method of constructing symplectic isotopies. Having its origin in Gromov's celebrated article [Gro], it

utilizes moduli spaces of pseudoholomorphic curves. One fixes a homotopy $h(t) := J_t$, $t \in [0,1]$, of ω-tame almost complex structures and considers the relative moduli space,

$$\mathscr{M}_h := \big\{ (C,t) : C \text{ is an imbedded } J_t\text{-holomorphic curve in the homology class } [A] \big\},$$

equipped with a natural topology and with the projection $\pi_h : (C,t) \in \mathscr{M}_h \mapsto t \in [0,1]$. It follows essentially from [Gro] that for a generic homotopy $h(t) = J_t$ the space $\mathscr{M}_h$ is a smooth manifold of expected dimension $\mathsf{dim}_{\mathbb{R}} \mathscr{M}_h = 2c_1(X,\omega)[A] + 2g - 1$ and the projection π_h is also smooth. Moreover, if this dimension is positive, then there exists a generic path $h(t) = J_t$ such that the original surface Σ_0 (resp. Σ_1) is J_0-holomorphic (resp. J_1-holomorphic) so that $(\Sigma_i, i) \in \mathscr{M}_h$ for $i = 0,1$. Using a trivial but crucial observation that for every $(C,t) \in \mathscr{M}_h$ the curve C is an ω-symplectic real surface, one tries to find the desired isotopy Σ_t by constructing a continuous section $\sigma : t \in [0,1] \mapsto (\Sigma_t, t) \in \mathscr{M}_h$ connecting $(\Sigma_0, 0)$ with $(\Sigma_1, 1)$.

One can easily see possible obstacles to the existence of such a section $\sigma : [0,1] \to \mathscr{M}_h$. The first one is that the projection $\pi_h : \mathscr{M}_h \to [0,1]$, considered as a function, can have local maxima and minima. Indeed, if some $(C^*, t^*) \in \mathscr{M}_h$ appears as a local maximum of π_h, then for all $t > t^*$ there exists no J_t-holomorphic curves sufficiently close to C^*. Observe that the mere fact of existence of a J_t-holomorphic curve C_t for $t > t^*$ does not help much, because this does not imply that such a curve C_t is symplectically isotopic to Σ_0 or Σ_1. Note that exactly the existence of J-holomorphic curves is the main technical tool in the Gromov's article [Gro].

On the other hand, this obstacle does not appear in the case when the projection π_h has the following *saddle point property*: the Hesse matrix of π_h at any critical point has at least one positive and at least one negative eigenvalues. In *Section 4* we prove

Theorem 2. *Assume that $c_1(X,\omega)[A] > 0$. Then for a generic homotopy $h(t)$ all critical points of the projection $\pi_h : \mathscr{M}_h \to [0,1]$ are saddle.*

On the other hand, we also show that in the case $c_1(X,\omega)[A] \leqslant 0$ local maxima and minima of the projection $\pi_h : \mathscr{M}_h \to [0,1]$ do exist in the case of an apropriate generic choice of h. Note that in the example of Fintushel and Stern [Fi-St] one has $c_1(X,\omega)[A] = 0$.

The next obstacle to existence of the desired section $\sigma : [0,1] \to \mathscr{M}_h$ comes from the fact that in general $\mathscr{M}_h$ is not compact and the projection π_h is not proper. This means that while attempting to construct the section $\sigma : [0,1] \to \mathscr{M}_h$ we go to "infinity" in the space $\mathscr{M}_h$. The Gromov compactness theorem provides control on the limiting behavior of curves C_t in this case. It says that some subsequence, say C_{t_n}, converges in a certain weak sense to a pseudoholomorphic curve C^* in the same homology class $[A]$. The curve C^* can have several irreducible components, some of them multiple, and also several singular points. Thus we come to a problem of describing symplectic isotopy classes of imbedded pseudoholomorphic curves close to a given singular curve C^*.

Here we have essentially two different difficulties. The first one comes from multiple components. At the moment, we have no remedy for this problem. So we appempt to avoid the appearence of multiple components. This is done by imposing the following additional constraint. We consider the curves C_t which pass through fixed points $\boldsymbol{x} = (x_1, \ldots, x_k)$ on X. This means, however, that now we must consider a new moduli space $\mathscr{M}_{h,\boldsymbol{x}}$ with a new projection $\pi_{h,\boldsymbol{x}} : \mathscr{M}_{h,\boldsymbol{x}} \to [0,1]$. In *Section 4* we also show that if the number k of the fixed points is strictly less than $c_1(X,\omega)[A]$, then the saddle point property for $\pi_{h,\boldsymbol{x}}$ remains valid. An easy calculation shows that fixing $k = 3d-1$ generic points on $\mathbb{CP}^2$, we

can avoid the appearance of multiple components in pseudoholomorphic curves of degree $d \leqslant 6$. This explains the restriction in the main theorem.

In the case when C^* has no multiple components we still have to describe the possible symplectic isotopy classes of imbedded pseudoholomorphic curves close to C^*. Recall that by the result of Micallef and White (see *Paragraph 1.2*) every singular point of C^* is isolated and topologically equivalent to a singular point of a usual holomorphic curve. In this way we come to the *local symplectic isotopy problem* which asks about possible symplectic isotopy types of imbedded pseudoholomorphic curves in a neighborhood of a given isolated singularity, see *Paragraph 6.2* for details.

The solution of the local symplectic isotopy problem is based on the simple observation that for *holomorphic curves* this problem has a trivial solution. Namely, a generic holomorphic deformation of a given (local) holomorphic curve C^* gives a non-singular curve C, and the set of such curves is open and connected. Using this fact, we expoit essentially the same method as in the case of the global problem and prove that it is possible to deform isotopically an imbedded pseudoholomorphic curve C sufficiently close to a given singular curve C^* into a genuine holomorphic curve. In this way we prove

Theorem 3. *There exists a unique symplectic isotopy class of non-singular pseudoholomorphic curves which are close to a given pseudoholomorphic curve C^* without multiple components.*

It should be noted that in the proof the saddle point property from *Theorem 2* is used in an essential way. *Theorem 1* follows now by the procedure of avoiding multiple components as it is explained above.

Further technical results of the paper are as follows. *Section 3* is devoted to the deformation problem of pseudoholomorphic maps with prescribed singularities. It is shown that the subspace of such maps is an immerced submanifold of expected codimension in the total moduli space of pseudoholomorphic maps. In *Section 4* the second variation of the $\overline{\partial}$-equation is computed. The result establishes the relationship between the geometry of a pseudoholomorphic curve C^* corresponding to a critical point of the projection $\pi_h : \mathscr{M}_h \to [0,1]$ and the eigenvalues of the Hesse matix $d^2\pi_h$ at this point. Combined with transversality results, this yields the proof of *Theorem 2*. Finally, in *Section 5* the problem of smoothing of nodal points on pseudoholomorphic curves is studied.

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1. Deformation and the normal sheaf of pseudoholomorphic curves

In this section we give a brief description of pseudoholomorphic curves and the related deformation theory.

1.1. **Pseudoholomorphic curves.**

First we collect some facts from the Gromov's theory. Since there are several books devoted to or treating this theme (see *e.g.* [McD-Sa-1] or [McD-Sa-2]) we only mention the basic definitions and results we shall use later.

Definition 1.1.1. An *almost complex structure* on a manifold X is an endomorphism $J \in TM$ of the tangent bundle such that $J^2 = -\mathsf{Id}$. The pair (X, J) is called an *almost complex manifold*. One of the most important classes of such manifolds appears in symplectic geometry. An almost complex structure on a symplectic manifold (X, ω) is called *ω-tame* if $\omega(v, Jv) > 0$ for any non-zero tangent vector v. It is well-known that the set

$\mathscr{J}_\omega$ of ω-tame almost complex structures is non-empty and contractible, (see *e.g.* [Gro], [McD-Sa-1], or [McD-Sa-2]). In particular, any two ω-tame almost complex structures J_0 and J_1 can be connected by a homotopy (path) $J_t, t \in [0,1]$, inside $\mathscr{J}_\omega$.

Definition 1.1.2. A *parameterized J-holomorphic curve* in an almost complex manifold (X,J) is given by a (connected) Riemann surface S with a complex structure J_S on S and a non-constant C^1-map $u : S \to X$ satisfying the Cauchy-Riemann equation

$$du + J \circ du \circ J_S = 0. \tag{1.1.1}$$

In this case we call u a (J_S, J)-holomorphic map, or simply J-holomorphic map. Here we use the fact that if u is not constant, then such a structure J_S is unique. We shall also use the notion *J-curve* which, depending on the context, will mean a map $u : S \to X$, *i.e.* a parameterized curve in X, or an image $u(S)$ of J-holomorphic map, taken with appropriate multiplicity, *i.e.* a *non-parameterized J-curve*.

The equation (1.1.1) is elliptic with the Cauchy-Riemann symbol. This provides regularity properties for u. In particular, u is Hölder $C^{l+1,\alpha}$-smooth, $u \in C^{l+1,\alpha}(S,X)$, provided $J \in C^{l,\alpha}$ with integer $l \geqslant 1$ and $0 < \alpha < 1$. To simplify the notations we set $\ell := l + \alpha$ and write C^ℓ to indicate $C^{l,\alpha}$-smoothness. In what follows we shall assume that almost complex structures J on X are C^ℓ-smooth for some fixed sufficiently big non-integer ℓ.

An easy consequence of the tameness condition is that any J-holomorphic *imbedding* $u : S \to X$ with $J \in \mathscr{J}_\omega$ is *symplectic* *i.e.* the pull-back $u^*\omega$ is non-degenerate on S. The converse is also true: Any $C^{\ell+1}$-smooth symplectic imbedding $u : S \to X$ with $\ell > 1$ is J-holomorphic for some C^ℓ-smooth ω-tame structure J. For *immersions* the situation is more complicated. We state a result in a setting which will be relevant later on (see, *e.g.* [Gro] or [McD-Sa-2] for details).

Lemma 1.1.1. *Let (X,ω) be a symplectic manifold with $\mathsf{dim}_{\mathbb{R}} X = 4$, and $u : S \to X$ an ω-symplectic C^1-map such that $u(S)$ has only simple transversal positive self-intersection points.*

Then there exist an ω-tame almost complex structure J on X and a complex structure J_S on S and making u a J-holomorphic map.

It is worth to make the following remark. If $x \in X$ is a self-intersection point of u, $x = u(z_1) = u(z_2)$ with $z_1 \neq z_2$, such that the tangent planes $du(T_{z_i}) \subset T_x X$ are transversal and *complex* with respect to some structure J_x in $T_x X$, then the intersection index of planes $du(T_{z_i})$ in x is *positive*. However, it is possible that two *symplectic* planes L_i in $(\mathbb{R}^4, \omega)$ have *negative* intersection index.

More detailed considerations lead to the *genus formula* (also called *adjunction formula*) for immersed symplectic surfaces in symplectic four-folds. For this let (X,ω) be a symplectic manifold of dimension 4, $S := \bigsqcup_{j=1}^d S_j$ a compact oriented surface and $u : S \to X$ an immersion with only transversal self-intersection points. Denote by g_j the genus of S_j, by $[C]$ the homology class of the image $C := u(S)$, $[C]^2$ the homological self-intersection number of $[C]$, and by $c_1(X)$ the first Chern class of (X,ω). Define the *geometric self-intersection number* δ of $M = u(S)$ as the algebraic number of pairs $z' \neq z'' \in S$ with $u(z') = u(z'')$, taken with the sign corresponding to the intersection index.

Lemma 1.1.2. *Suppose that $u : S \to X$ is a symplectic immersion which is compatible with the orientation on each component S_j of S. Then*

$$\sum_{j=1}^{d} g_j = \frac{[C]^2 - c_1(X)[C]}{2} + d - \delta. \tag{1.1.2}$$

An elementary proof uses the fact that for a *symplectic* immersion $u : S \to (X,\omega)$ one has $c_1(X)[C] = \chi(S) + \chi(N)$, where N is the normal bundle and χ denotes the Euler characteristic. Finally, one observes that $\chi(N) = [C]^2 - 2\delta$. For details, see [Iv-Sh-1].

1.2. **Local structure of pseudoholomorphic curves.** For the most results of this paragraph we refer to [Mi-Wh] where a very precise description of the local structure of pseudoholomorphic curves is given. As a rough summary, one can say that the local behavior of (non-parameterized) pseudoholomorphic curves is essentially the same as for usual holomorphic curves.

Lemma 1.2.1. ([Mi-Wh]) *Let (X,J) be an almost complex manifold of $\mathsf{dim}_{\mathbb{C}} X = n$, $u : S \to X$ a J-holomorphic map, and $x \in X$ a point. Suppose that $J \in C^2$ and that for any $x' \in X$ sufficiently close to x the pre-image $u^{-1}(x)$ is finite. Then there exist neighborhoods $U \subset X$ of x, $U' \subset \mathbb{C}^n$ of $0 \in \mathbb{C}^n$ and a C^1-diffeomorphism $\varphi : U \to U'$ such that $C' := \varphi\big(u(S) \cap U\big)$ is a proper analytic curve in U' and such that $\varphi_*(J_x) = J_{\mathsf{st}}$, where J_{st} is the standard complex structure in $\mathbb{C}^n$.*

In particular, the notion of a (local) irreducible component of a J-holomorphic curve $C = u(S)$ is well-defined. Further, in the special case when (X,J) is an *almost complex surface* one can correctly define
i) the intersection index $\delta_{ij}(x) \in \mathbb{N}$ of two local components C_i and C_j at $x \in X$, and
ii) the nodal number $\delta_i(x) \in \mathbb{N}$ of a local component C_i at $x \in X$ (see [Mil], § 10 and *Definition 6.2.1*).

The main properties of these local invariants are summarized in

Lemma 1.2.2. *i) If $x \in C_i \cap C_j$, then $\delta_{ij}(x) \geqslant 1$. The equality holds if and only if C_i and C_j are smooth and intersect transversally in x;*

ii) The set $\{z \in S : \delta_i(u(z)) > 0\}$ is discrete in S;

iii) Suppose additionally that $S = \sqcup_{j=1}^{d} S_j$ is a closed surface and $u : S \to X$ is an imbedding almost everywhere on S. Set $C := u(S)$. Denote by δ the sum of all local intersection indices $\delta_{ij}(x)$ and all local nodal numbers $\delta_i(x)$, the homology class of C by $[C]$, and the genera of particular components S_j by g_j. Then

$$\textstyle\sum_{j=1}^{d} g_j = \frac{[C]^2 - c_1(X)[C]}{2} + d - \delta. \tag{1.2.1}$$

The formula (1.2.1) is the *genus formula for pseudoholomorphic curves*. We shall also apply a local version of this result. Here we say that a pseudoholomorphic curve C in a symplectic manifold X is *parameterized by a real surface* S if there exists a map $u : S \to X$ which is an imbedding outside a discrete subset in S. Such a surface S, possibly not connected, can be constructed as the normalization of $C \subset X$.

Lemma 1.2.3. *Let $B \subset \mathbb{R}^4$ be the unit ball equipped with the standard symplectic structure ω_{st}, and C_1, C_2 pseudoholomorphic curves in B. Assume that the boundaries of the curves ∂C_i are imbedded in the boundary of the ball ∂B, are sufficiently close to each other, and*

that every C_i meets transversally ∂B. Denote by χ_i the Euler characteristic of the surface S_i parameterizing C_i and by δ_i the sum of the nodal number of singular points of C_i. Then

$$\chi_1 - 2\delta_1 = \chi_2 - 2\delta_2 \tag{1.2.2}$$

Proof. It is shown in [Iv-Sh-1] that every C_i can be perturbed to a nearby pseudoholomorphic curve C_i' in such a way that every singular point $x \in C_i$ with the nodal number $\delta_x(C_i) \geqslant 2$ "splits" into $\delta_x(C_i)$ *nodal points* on the perturbed curve C_i', *i.e.* the points where exactly two branches of C_i' meet transversally. By this procedure the topology of each S_i and the whole nodal number of every C_i remain unchanged. After this, one can replace a sufficiently small neighborhood of every nodal point $x \in C_i'$ by a symplectically imbedded handle. This "symplectic surgery of C_i'" produce *imbedded* pseudoholomorphic curves C_i'' with $\chi(C_i'') = \chi_i - 2\delta_i$.

Moreover, all this can be carried out with the boundaries ∂C_i unchanged. Further, the hypothesis of the lemma implies that the boundaries ∂C_i are transversal to the standard symplectic structure on $\partial B = S^3$ and are isotopic as *transversal links*, see [Iv-Sh-1] and [Eli]. Now one applies the theorem of Bennequin [Bn], see also [Eli], which claims that, up to sign convention, $\chi(C_i'')$ is the *Bennequin index of* ∂C_i and depends only on the *transversal isotopy* class of ∂C_i. The lemma follows. $\square$

The result of Micallef and White, *Lemma 1.2.1*, is not sufficient for our purpose, because it does not allow us to control local structure of pseudoholomorphic curves under deformation. A necessary tool is provided by the following statement proven in [Iv-Sh-1]. Here and thereafter Δ denotes the unit disc in $\mathbb{C}$ equipped with the standard complex structure.

Lemma 1.2.4. *Suppose that a $f \in L^{1,2}_{\mathsf{loc}}(\Delta, \mathbb{C}^n)$ is not identically 0 and satisfies a.e. the inequality*

$$|\bar\partial f(z)| \leqslant h(z) \cdot |z|^k \cdot |f(z)| \tag{1.2.3}$$

for some $k \in \mathbb{N}$ and nonnegative $h \in L^p_{\mathsf{loc}}(\Delta)$ with $2 < p < \infty$. Then

$$f(z) = z^\mu \big(P^{(k)}(z) + z^k g(z) \big), \tag{1.2.4}$$

where $\mu \in \mathbb{N}$, $P^{(k)}$ is a polynomial in z of degree $\leqslant k$ with $P^{(k)}(0) \neq 0$, and $g \in L^{1,p}_{\mathsf{loc}}(\Delta, \mathbb{C}^n) \hookrightarrow C^{0,\alpha}$, $\alpha = 1 - \frac{2}{p}$, with $g(0) = 0$.

Using this result one can obtain the following description of the local behavior of a pseudoholomorphic map. Note that on a given almost complex manifold (X, J) in a neighborhood of a given point $x_0 \in X$ there exist an (integrable) complex structure J^* with $J^*(x_0) = J(x_0)$ and J^*-holomorphic coordinates $w_1, \dots, w_n$, $n = \mathsf{dim}_{\mathbb{C}} X$.

Lemma 1.2.5. *i) Assume that J is C^1-smooth and $u : \Delta \to X$ is a non-constant J-holomorphic map with $u(0) = x_0$. Then in coordinates $w_1, \dots, w_n$ chosen as above in a neighborhood of $x_0 \in X$ the map u has the form*

$$u(z) = z^\mu \cdot P^{(\mu-1)}(z) + z^{2\mu-1} \cdot v(z), \tag{1.2.5}$$

where $\mu \in \mathbb{N}$, $P^{(\mu-1)}(z)$ is a complex $\mathbb{C}^n$-valued polynomial of degree $\leqslant \mu - 1$ with $P^{(\mu-1)}(0) \neq 0$, and $v(z) \in L^{1,p}(\Delta, \mathbb{C}^n)$ with $v(0) = 0$.

ii) Assume that J is C^1-smooth and let $u_1, u_2 : \Delta \to X$ be J-holomorphic maps such that u_1 is an immersion and $u_2(0) \in u_1(\Delta)$. Then there exists $r \in]0, 1[$ such that either $u_2(\Delta(r)) \subset u_1(\Delta)$ or $u_2(\Delta(r)) \cap u_1(\Delta) = u_2(0)$.

iii) Let J be a C^ℓ-smooth almost complex structure on the ball $B \subset \mathbb{C}^n$ with $J(0) = J_{\mathsf{st}}(0)$, and let $u_1, u_2 : \Delta \to B$ be J-holomorphic maps with $u_1(0) = u_2(0) = 0 \in B$, such that $u_1 \not\equiv u_2$.

Then there exists a uniquely defined $\nu \in \mathbb{N}$ and $w(z) \in C^1(\Delta, \mathbb{C}^n)$ such that

$$u_1(z) - u_2(z) = z^\nu w(z). \tag{1.2.6}$$

Proof. The first and second parts of the lemma are proven in [Iv-Sh-1]. For the third see *Remark 1.6* and *Section 6* of [Mi-Wh].

Definition 1.2.1. If for an appropriate local complex coordinate z on S a J-holomorphic map $u : S \to X$ has the form (1.2.5), then we call the (uniquely defined) μ the *multiplicity of u at the point $z = 0$*.

Definition 1.2.2. A J-holomorphic map $u : S \to X$ is *multiple* if there exists a non-empty $U \subset S$ such that the restriction $u|_U$ can be represented as a composition $u|_U = u' \circ \varphi$ where $u' : \Delta \to X$ is a J-holomorphic map and $\varphi : U \to \Delta$ is a (branched) covering of degree $m \geqslant 2$. In other words, u is multiple if some part of the image $u(S)$ is multiply covered by u.

1.3. **Deformation of pseudoholomorphic maps and the Gromov operator $D_{u,J}$.** Roughly speaking, the main idea of the Gromov's theory is to construct and study J-holomorphic curves in a symplectic manifold (X, ω) for some special (*e.g. integrable*) J. Often, one can show the existence of a J_0-holomorphic curve with some other almost complex structure J_0, see *e.g. Lemma 1.1.1.* If both J and J_0 are ω-tame, then there exists a homotopy $\{J_t\}_{t\in[0,1]}$ from J_0 to $J_1 = J$. Hence one could try to deform the constructed J_0-holomorphic map $u_0 : S \to X$ into a J_1-holomorphic one using the continuity principle. The first step in this direction is to study the linearization of (*i.e.* the first variation) the equation (1.1.1). This means that we are interested in the first differential of the section $\sigma_{\bar\partial}$.

Fix a compact surface S of genus g. Denote by $\mathscr{J}_S$ the Banach manifold of $C^{1,\alpha}$-smooth complex structures on S with some fixed $\alpha \in]0,1[$. Thus

$$\mathscr{J}_S = \{J_S \in C^{1,\alpha}(S, \mathsf{End}(TS)) : J_S^2 = -\mathsf{Id}\} \tag{1.3.1}$$

and the tangent space to $\mathscr{J}_S$ at J_S is

$$T_{J_S}\mathscr{J}_S = \{I \in C^{1,\alpha}(S, \mathsf{End}(TS)) : J_S I + I J_S = 0\} \equiv C^{1,\alpha}(S, \Lambda^{0,1}S \otimes TS), \tag{1.3.2}$$

where $\Lambda^{0,1}S$ denotes the line bundle of $(0,1)$-form on S.

Let $\mathscr{J}$ be an open *connected* subset in the Banach manifold of all C^ℓ-smooth almost complex structures on X for some fixed non-integer $\ell > 2$. In our context the most interesting example is the set $\mathscr{J}_\omega$ of C^ℓ-smooth ω-tame almost complex structures on X. The tangent space to $\mathscr{J}$ at J consists of C^ℓ-smooth J-antilinear endomorphisms of TX,

$$T_J\mathscr{J} = \{I \in C^\ell(X, \mathsf{End}(TX)) : JI + IJ = 0\} \equiv C^\ell(X, \Lambda^{0,1}X \otimes TX), \tag{1.3.3}$$

where $\Lambda^{0,1}X$ denotes the complex bundle of $(0,1)$-forms on X.

Fix p with $2 < p < \infty$. Then the set $L^{1,p}(S, X)$ of all Sobolev $L^{1,p}$-smooth maps from S to X is a Banach manifold. For $u \in L^{1,p}(S, X)$ we denote

$$E_u := u^*TX. \tag{1.3.4}$$

In this notation, the tangent space at $u \in L^{1,p}(S, X)$ is $T_u L^{1,p}(S, X) = L^{1,p}(S, E_u)$, the space of $L^{1,p}$-smooth sections of the pulled-back tangent bundle of X.

Fix a homology class $[C] \in \mathsf{H}_2(X,\mathbb{Z})$ and consider the set

$$\mathscr{S} = \{u \in L^{1,p}(S,X) : u(S) \in [C]\} \tag{1.3.5}$$

of maps u representing the class $[C]$. Then $\mathscr{S}$ is open in $L^{1,p}(S,X)$ and has the same tangent space, $T_u\mathscr{S} = L^{1,p}(S,E_u)$. Since $\mathscr{J}$ is connected, the first Chern class $c_1(X,J)$ is constant on $\mathscr{J}$. We shall denote it simply by $c_1(X)$. Set

$$\mu := \langle c_1(X),[C]\rangle. \tag{1.3.6}$$

Consider the subset $\mathscr{P} \subset \mathscr{S} \times \mathscr{J}_S \times \mathscr{J}$ consisting of all triples (u,J_S,J) with u being (J_S,J)-holomorphic,

$$\mathscr{P} = \{(u,J_S,J) \in \mathscr{S} \times \mathscr{J}_S \times \mathscr{J} : du + J\circ du\circ J_S = 0\}. \tag{1.3.7}$$

Let ∇ be some symmetric connection on TX. Covariant differentiation of (1.1.1) gives the equation for the tangent space to $\mathscr{P}$. Namely, a vector $(v,\dot J_S,\dot J)$ is tangent to $\mathscr{P}$ at the point (u,J_S,J) if it satisfies the equation

$$\nabla v + J\circ\nabla v\circ J_S + (\nabla_v J)\circ(du\circ J_S) + J\circ du\circ \dot J_S + \dot J\circ du\circ J_S = 0. \tag{1.3.8}$$

Definition 1.3.1. a) For a complex bundle E over S let

$$L^p_{(0,1)}(S,E) := L^p(S,E\otimes\Lambda^{(0,1)}S) \tag{1.3.9}$$

denote the Banach space of L^p-integrable E-valued (0,1)-forms on S.

b) Let u be a J-holomorphic curve in X. Define the operator $D_{u,J} : L^{1,p}(S,E_u) \to L^p_{(0,1)}(S,E_u)$ as

$$D_{u,J}(v) := \nabla v + J\circ\nabla v\circ J_S + (\nabla_v J)\circ du\circ J_S \tag{1.3.10}$$

c) Define complex Banach bundles $\mathscr{E}$ and $\mathscr{E}'$ over $\mathscr{S}\times\mathscr{J}_S\times\mathscr{J}$ by

$$\mathscr{E}_{(u,J_S,J)} := L^{1,p}(S,E_u) \quad\text{and}\quad \mathscr{E}'_{(u,J_S,J)} := L^p_{(0,1)}(S,E_u). \tag{1.3.11}$$

These bundles are C^ℓ-smooth and the formula (1.3.10) defines a $\mathbb{R}$-linear homomorphism $D = D_{u,J_S,J} : \mathscr{E} \to \mathscr{E}'$ which is $C^{\ell-1}$-smooth. The bundle $\mathscr{E}$ is essentially the tangent bundle to $\mathscr{S}$, whereas $\mathscr{E}'$ appears as the space where the equation (1.1.1) "lives". More precisely, (1.1.1) defines a section $\sigma_{\bar\partial}$ of $\mathscr{E}'$,

$$\sigma_{\bar\partial} : (u,J_S,J) \in \mathscr{S}\times\mathscr{J}_S\times\mathscr{J} \;\mapsto\; (du + J\circ du\circ J_S) \in \mathscr{E}'_{(u,J_S,J)}, \tag{1.3.12}$$

such that the equation (1.1.1) reads $\sigma_{\bar\partial}(u,J_S,J) = 0$. The space $\mathscr{P}$ appears then as the zero set of the section $\sigma_{\bar\partial}$.

Remark. Here and thereafter we use the normalization $\frac{\partial}{\partial\bar z} = \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}$, deviating from the usual convention $\frac{\partial}{\partial\bar z} = \frac{\mathbf{1}}{\mathbf{2}}\cdot(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y})$. The same normalization is used for all operators with Cauchy-Riemann symbol.

Lemma 1.3.1. *Let $\mathscr{X}$ be a Banach manifold, $\mathscr{E}\to\mathscr{X}$ and $\mathscr{E}'\to\mathscr{X}$ C^1-smooth Banach bundles over $\mathscr{X}$, ∇ and ∇' linear connections in $\mathscr{E}$ and $\mathscr{E}'$ respectively, σ a C^1-smooth section of $\mathscr{E}$, and $D : \mathscr{E}\to\mathscr{E}'$ a C^1-smooth bundle homomorphism.*

i) If $\sigma(x) = 0$ for some $x\in\mathscr{X}$, then the map $\nabla\sigma_x : T_x\mathscr{X}\to\mathscr{E}_x$ is independent of the choice of the connection ∇ in $\mathscr{E}$;

ii) Set $K_x := \mathsf{Ker}\,(D_x : \mathscr{E}_x \to \mathscr{E}'_x)$ *and* $Q_x := \mathsf{Coker}\,(D_x : \mathscr{E}_x \to \mathscr{E}'_x)$ *with the corresponding imbedding* $i_x : K_x \to \mathscr{E}_x$ *and projection* $p_x : \mathscr{E}'_x \to Q_x$. *Let* ∇^{Hom} *be the connection in* $\mathsf{Hom}\,(\mathscr{E},\mathscr{E}')$ *induced by the connections* ∇ *and* ∇'. *Then the map*

$$p_x \circ (\nabla^{\mathsf{Hom}} D_x) \circ i_x : T_x\mathscr{X} \to \mathsf{Hom}\,(K_x, Q_x) \tag{1.3.13}$$

is independent of the choice of connections ∇ *and* ∇'.

Remark. Taking this lemma into account, we shall use the following notation. For $\sigma \in \Gamma(\mathscr{X},\mathscr{E})$, $D \in \Gamma(\mathscr{X},\mathsf{Hom}\,(\mathscr{E},\mathscr{E}'))$ and $x \in \mathscr{X}$ as in the hypothesis of the lemma we shall denote by $\nabla\sigma_x : T_x\mathscr{X} \to \mathscr{E}_x$ and $\nabla D : T_x\mathscr{X} \times \mathsf{Ker}\,D_x \to \mathsf{Coker}\,D_x$ the corresponding operators without pointing out which connections were used to define them.

Proof. *i)* Let $\widetilde{\nabla}$ be another connection in $\mathscr{E}$. Then $\widetilde{\nabla}$ has the form $\widetilde{\nabla} = \nabla + A$ for some $A \in \Gamma(\mathscr{X},\mathsf{Hom}\,(T\mathscr{X},\mathsf{End}(\mathscr{E})))$. So for $\xi \in T_x\mathscr{X}$ we obtain $\widetilde{\nabla}_\xi\sigma - \nabla_\xi\sigma = A(\xi,\sigma(x)) = 0$.

ii) Similarly, let $\widetilde{\nabla}'$ be another connection in $\mathscr{E}'$, and let $\widetilde{\nabla}^{\mathsf{Hom}}$ be the connection in $\mathsf{Hom}\,(\mathscr{E},\mathscr{E}')$ induced by $\widetilde{\nabla}$ and $\widetilde{\nabla}'$. Then $\widetilde{\nabla}'$ also has the form $\widetilde{\nabla} = \nabla + A'$ for some $A' \in \Gamma(\mathscr{X},\mathsf{Hom}\,(T\mathscr{X},\mathsf{End}(\mathscr{E}')))$. So for $\xi \in T_x\mathscr{X}$ we obtain $\widetilde{\nabla}^{\mathsf{Hom}}_\xi D - \nabla^{\mathsf{Hom}}_\xi D = A'(\xi)\circ D_x - D_x \circ A(\xi)$. The statement of the lemma now follows from the identities $p_x \circ D_x = 0$ and $D_x \circ i_x = 0$. □

Remark. The operator $D_{u,J}$ is the linearization of the equation (1.1.1). Thus *Lemma 1.3.1* shows that the definition of $D_{u,J}$ is independent of the choice of ∇. In particular, one can also use non-symmetric connections, *e.g.* those compatible with J, as it is done in [Gro]. However, it is convenient to have a fixed connection considering varying almost complex structures J on X. But this is impossible for ∇ compatible with J. On the other hand, with a symmetric connection computations become simpler.

The operator $D_{u,J}$, as well as the equation (1.1.1) itself, is elliptic of order 1 with the Cauchy-Riemann symbol. This implies standard regularity properties for $D_{u,J}$. In particular, the kernel and the cokernel are of finite dimension. The Riemann-Roch formula gives the index of $D_{u,J}$:

$$\mathsf{dim}_{\mathbb{R}}\mathsf{Ker}\,D_{u,J} - \mathsf{dim}_{\mathbb{R}}\mathsf{Coker}\,D_{u,J} = 2\cdot\big(\mu + n(1-g)\big), \tag{1.3.14}$$

where $\mu := c_1(X)\cdot[u(S)]$, g is the genus of S, and n the *complex* dimension of X, *i.e.* $n := \frac{1}{2}\mathsf{dim}_{\mathbb{R}}X$. The factor 2 appears because we compute *real, not complex* dimensions of the (co)kernel.

1.4. **Holomorphic structure on the induced bundle.** Now we want to understand the structure of the operator $D_{u,J}$ in more detail. Note that the pulled-back bundle $E_u = u^*TX$ carries a complex structure, namely J itself, or more accurately u^*J. However, the operator $D := D_{u,J}$ is only $\mathbb{R}$-linear. So we decompose it into J-linear and J-antilinear parts. Namely, for $v \in L^{1,p}(S,E)$ we write $Dv = \frac{1}{2}\big(Dv - JD(Jv)\big) + \frac{1}{2}\big(Dv + JD(Jv)\big) = \bar\partial_{u,J}v + R(v)$.

Definition 1.4.1. The J-linear part $\bar\partial_{u,J}$ of the operator $D_{u,J}$ is called the *$\bar\partial$-operator associated with a J-holomorphic map u.*

By the definition, the operator $\bar\partial_{u,J} : L^{1,p}(S,E_u) \to L^p_{(0,1)}(S,E_u)$ is J-linear. The following statement is well known in the smooth case.

Lemma 1.4.1. *Let S be a Riemann surface with a complex structure J_S and E a $L^{1,p}$-smooth complex vector bundle of rank r over S. Let also $\overline{\partial}_E : L^{1,p}(S,E) \to L^p_{(0,1)}(S,E)$ be a complex linear differential operator satisfying the condition*

$$\overline{\partial}_E(f\xi) = \overline{\partial}_S f \otimes \xi + f \cdot \overline{\partial}_E \xi, \tag{1.4.1}$$

where $\overline{\partial}_S$ is the Cauchy-Riemann operator on S associated to J_S. Then the sheaf

$$U \subset S \mapsto \mathscr{O}(E)(U) := \{\, \xi \in L^{1,p}(U,E) : \overline{\partial}_E \xi = 0 \,\} \tag{1.4.2}$$

is coherent and locally free of rank r. This defines a holomorphic structure on E for which $\overline{\partial}_E$ is the associated Cauchy-Riemann operator.

Remark. The condition (1.4.1) means that $\overline{\partial}_E$ is of order 1 and has the Cauchy-Riemann symbol. For the proof we refer to [Iv-Sh-1] and [Iv-Sh-2] for the general case, or to [H-L-S] for the case of line bundles.

Thus, according to *Lemma 1.4.1*, the operator $\overline{\partial}_{u,J}$ defines a holomorphic structure on the bundle E_u. We shall denote by $\mathscr{O}(E_u)$ the sheaf of holomorphic sections of E_u. The tangent bundle TS to our Riemann surface also carries a natural holomorphic structure. We shall denote by $\mathscr{O}(TS)$ the corresponding coherent sheaf.

Denote by $N_J(v,w)$ the Nijenhuis torsion tensor of the almost complex structure J, (see *e.g.* [Ko-No], Vol.II., p.123.) [1]

Lemma 1.4.2. *i) The J-antilinear part R of $D_{u,J}$ is related to u and J by the formula*

$$R(v)(\xi) = N_J(v, du(\xi)) \qquad \xi \in TS. \tag{1.4.3}$$

Thus R is a continuous J-antilinear operator from E to $\Lambda^{0,1}S \otimes E_u$ of order zero which satisfies $R \circ du \equiv 0$, i.e. $R(du(\eta),\xi) = 0$ for all $\eta, \xi \in TS$.

ii) If u is non-constant, then du defines an injective analytic morphism of coherent sheaves

$$0 \longrightarrow \mathscr{O}(TS) \xrightarrow{du} \mathscr{O}(E_u). \tag{1.4.4}$$

Proof. *i)* Formula (1.4.3) can be found in [McD-2]. The rest of part *i)* follows from the well-known fact that $N_J(v,w)$ is skew-symmetric and J-antilinear in both arguments.
ii) The fact that $du : TS \to E_u$ defines a morphism between coherent sheaves $\mathscr{O}(TS)$ and $\mathscr{O}(E_u)$ means that du is a *holomorphic* section of $T^*S \otimes E_u$. This is equivalent to relation

$$du \circ \overline{\partial}_S = \overline{\partial}_{u,J} \circ du. \tag{1.4.5}$$

For the proof of this fact we refer to [Iv-Sh-1] and [Iv-Sh-2].

Injectivity of the sheaf homomorphism du is equivalent to its nondegeneracy which is the case in our context. □

The zeros of the analytic morphism $du : \mathscr{O}(TS) \to \mathscr{O}(E_u)$ are isolated. So we obtain

Corollary 1.4.3. ([Mi-Wh], [Sk-1]) *The set of critical points of a J-holomorphic map is discrete, provided J is of class C^1.*

Definition 1.4.2. By the *order of zero* $\mathsf{ord}_p du$ of the differential du at a point $p \in S$ we shall understand the order of vanishing at p of the holomorphic morphism $du : \mathscr{O}(TS) \to \mathscr{O}(E_u)$.

It follows from *Lemma 1.4.2* that $\mathsf{ord}_p du$ is a well-defined non-negative integer.

[1] Note that in [Ko-No] another normalization constant is used. However, this is not essential for our purpose.

1.5. **The normal sheaf of a pseudoholomorphic curve.** From (1.4.4) we obtain the following short exact sequence of coherent sheaves

$$0 \longrightarrow \mathscr{O}(TS) \xrightarrow{du} \mathscr{O}(E_u) \longrightarrow \mathscr{N}_u \longrightarrow 0, \tag{1.5.1}$$

where $\mathscr{N}_u := \mathscr{O}(E)/du(\mathscr{O}(TS))$ is the quotient sheaf. It follows from *Lemma 1.4.2 ii)* that there is a decomposition $\mathscr{N}_u = \mathscr{O}(N_u) \oplus \mathscr{N}_u^{\mathsf{sing}}$ where N_u is a holomorphic vector bundle and $\mathscr{N}_u^{\mathsf{sing}} = \bigoplus_{z\in S} \mathbb{C}_z^{\mathsf{ord}_z du}$ is a discrete sheaf with support in the set of critical points a_i of u with the stalk $\mathbb{C}^{n_i}$ of dimension $n_i := \mathsf{ord}_{a_i} du$ at every such point a_i.

Definition 1.5.1. The quotient sheaf $\mathscr{N}_u := \mathscr{O}(E)/du(TS)$ is called the *normal sheaf of a J-curve* $u : S \to X$, N_u the *normal bundle to the J-curve* $u : S \to X$, and $[A] := \sum n_i[a_i]$ the *branching divisor of the J-curve* $u : S \to X$.

Denote by $\mathscr{O}([A])$ the sheaf of meromorphic functions on S having poles of order at most n_i at a_i. Then (1.5.1) gives rise to the exact sequence of coherent sheaves

$$0 \longrightarrow \mathscr{O}(TS) \otimes \mathscr{O}([A]) \xrightarrow{du} \mathscr{O}(E_u) \longrightarrow \mathscr{O}(N_u) \longrightarrow 0. \tag{1.5.2}$$

The holomorphic structure in N_u defines the Cauchy-Riemann operator $\overline{\partial}_N : L^{1,p}(S, N_u) \longrightarrow L^p_{(0,1)}(S, N_u)$. *Lemma 1.4.2* implies that the homomorphism $R : E_u \to E_u \otimes \Lambda^{(0,1)}S$ induces a J-antilinear bundle homomorphism $R_N : N_u \to N_u \otimes \Lambda^{(0,1)}S$. Define the operator

$$D^N_{u,J} : L^{1,p}(S, N_u) \longrightarrow L^p_{(0,1)}(S, N_u) \quad \text{by} \quad D^N_{u,J} := \overline{\partial}_N + R_N. \tag{1.5.3}$$

Definition 1.5.2. Let E be a holomorphic vector bundle over a compact Riemann surface S of genus g and let $D : L^{1,p}(S, E) \to L^p(S, \Lambda^{0,1}S \otimes E)$ be an operator of the form $D = \overline{\partial} + R$ where $R \in L^p\big(S, \mathsf{Hom}_{\mathbb{R}}(E, \Lambda^{0,1}S \otimes E)\big)$ with $2 < p < \infty$. Define $\mathsf{H}^0_D(S, E) := \mathsf{Ker}\, D$ and $\mathsf{H}^1_D(S, E) := \mathsf{Coker}\, D$. The groups $\mathsf{H}^i_D(S, E)$ are referred to as *D-cohomology groups of E*.

The Riemann-Roch formula gives the *index* of D,

$$\mathsf{ind}_{\mathbb{R}} D := \mathsf{dim}_{\mathbb{R}} \mathsf{H}^0_D(S, E) - \mathsf{dim}_{\mathbb{R}} \mathsf{H}^1_D(S, E) = 2\big(c_1(E) + \mathsf{rank}(E)(1-g)\big). \tag{1.5.4}$$

Remark. Taking into account the elliptic regularity of the Cauchy-Riemann operator, for given S, E and $R \in L^p$, $2 < p < \infty$, one can define $\mathsf{H}^i_D(S, E)$ as the (co)kernel of the operator $\overline{\partial} + R : L^{1,q}(S, E) \to L^q(S, \Lambda^{0,1}S \otimes E)$ for any $q \in]1, p]$. So the definition is independent of the choice of the functional spaces. Note also that the $\mathsf{H}^i_D(S, E)$ are of finite dimension provided that S is closed. For details, see *e.g.* [Iv-Sh-1].

The following lemmas contain main properties of D-cohomologies which will be used later. For complete proofs we refer to [Iv-Sh-1] and [Iv-Sh-2].

Lemma 1.5.1. *(Serre duality for D-cohomologies.) Let E be a holomorphic vector bundle over a compact Riemann surface S and let $D : L^{1,p}(S, E) \to L^p_{(0,1)}(S, E)$ be an operator of the form $D = \overline{\partial} + R$, where $R \in L^p\big(S, \mathsf{Hom}_{\mathbb{R}}(E, \Lambda^{0,1}S \otimes E)\big)$ with $2 < p < \infty$. Let $K_S := \Lambda^{1,0}S$ be the canonical holomorphic line bundle of S. Then there exists the naturally defined operator*

$$D^* = \overline{\partial} - R^* : L^{1,p}(S, E^* \otimes K_S) \to L^p_{(0,1)}(S, E^* \otimes K_S) \tag{1.5.5}$$

with $R^* \in L^p\big(S, \mathsf{Hom}_{\mathbb{R}}(E^* \otimes K_S, \Lambda^{0,1}S \otimes E^* \otimes K_S)\big)$ and the natural isomorphisms

$$\begin{aligned} \mathsf{H}^0_D(S, E)^* &\cong \mathsf{H}^1_{D^*}(S, E^* \otimes K_S), \\ \mathsf{H}^1_D(S, E)^* &\cong \mathsf{H}^0_{D^*}(S, E^* \otimes K_S), \end{aligned} \tag{1.5.6}$$

induced by the pairings

$$\begin{aligned} &\varphi\in \mathsf{H}^0_D(S,E), \quad \psi\in L^p_{(0,1)}(S,E^*\otimes K_S) &&\mapsto \langle\varphi,\psi\rangle &&:= \mathsf{Re}\textstyle\int_S \psi\circ\varphi \\ &\psi\in \mathsf{H}^0_D(S,E^*\otimes K_S), \quad \varphi\in L^p_{(0,1)}(S,E) &&\mapsto \langle\varphi,\psi\rangle &&:= \mathsf{Re}\textstyle\int_S \psi\circ\varphi \end{aligned} \tag{1.5.7}$$

If, in addition, R is $\mathbb{C}$-antilinear, then R^* is also $\mathbb{C}$-antilinear.

Remark. The lemma expresses the well-known relation $\mathsf{Ker}\,(D^*)=(\mathsf{Im}\,D)^\perp$ between a linear operator D and its adjoint. It is worth observing that the spaces themselves and the duality are defined only over the real numbers $\mathbb{R}$ and not over $\mathbb{C}$.

Lemma 1.5.2. ([H-L-S], *Vanishing theorem for D-cohomologies.*) *Let S be a closed Riemann surface of genus g and L a holomorphic line bundle over S, equipped with a differential operator $D=\bar\partial+R$ with $R\in L^p\big(S,\mathsf{Hom}_{\mathbb{R}}(L,\Lambda^{0,1}S\otimes L)\big)$, $p>2$. If $c_1(L)<0$, then $\mathsf{H}^0_D(S,L)=0$. If $c_1(L)>2g-2$, then $\mathsf{H}^1_D(S,L)=0$.*

The importance of the operator $D=\bar\partial+R$ lies in the fact that we can associate with the short exact sequence (1.5.1) the long exact sequence of D-cohomologies. Note, that due to *Lemma 1.4.2* we obtain the short exact sequence of complexes

$$\begin{array}{ccccccccc} 0 \longrightarrow & L^{1,p}(S,TS) & \xrightarrow{du} & L^{1,p}(S,E) & \xrightarrow{\mathsf{pr}} & L^{1,p}(S,E)\,/\,du(L^{1,p}(S,TS)) & \longrightarrow 0 \\ & \Big\downarrow{\scriptstyle\bar\partial_S} & & \Big\downarrow{\scriptstyle D} & & \Big\downarrow{\scriptstyle\overline{D}} & \\ 0 \longrightarrow & L^p_{(0,1)}(S,TS) & \xrightarrow{du} & L^p_{(0,1)}(S,E) & \xrightarrow{\mathsf{pr}} & L^p_{(0,1)}(S,E)\,/\,du(L^p_{(0,1)}(S,TS)) & \longrightarrow 0 \end{array} \tag{1.5.8}$$

where $\overline{D}$ is induced by $D\equiv D_{u,J}$.

Lemma 1.5.3. *For $\overline{D}$ as in* (1.5.8), $\mathsf{Ker}\,\overline{D}=\mathsf{H}^0_D(S,N_u)\oplus\mathsf{H}^0(S,\mathscr{N}_u^{\mathsf{sing}})$ *and* $\mathsf{Coker}\,\overline{D}=\mathsf{H}^1_D(S,N_u)$.

Proof. For an open set $U\subset S$ let $\Gamma_D(U,E_u):=\{v\in L^{1,p}_{\mathsf{loc}}(U,E_u)\ :\ Dv=0\}$. Use the analogous notation for N_u. Consider the sheaves $U\mapsto\Gamma(U,\mathscr{O}(TS))$, $U\mapsto\Gamma_D(U,E_u)$, and $U\mapsto\Gamma_D(U,N_u)\oplus\Gamma(U,\mathscr{N}_u^{\mathsf{sing}})$. It is easy to show that the first two columns of the diagram (1.5.8) define fine resolutions of the sheaves $\mathscr{O}(TS)$ and $\Gamma_D(\,\cdot\,,E_u)$. Moreover, du defines injective homomorphisms between these sheaves and between their resolutions. An explicit computation shows that $\Gamma_D(\,\cdot\,,N_u)\oplus\Gamma(\,\cdot\,,\mathscr{N}_u^{\mathsf{sing}})$ is the corresponding quotient sheaf and that the third column of (1.5.8) is its resolution. For details, see [Iv-Sh-1]. ☐

Corollary 1.5.4. *The short exact sequence* (1.5.1) *induces the long exact sequence of D-cohomologies*

$$\begin{array}{cccccccccc} 0 & \longrightarrow & \mathsf{H}^0(S,TS) & \longrightarrow & \mathsf{H}^0_D(S,E) & \longrightarrow & \mathsf{H}^0_D(S,N_u)\oplus\mathsf{H}^0(S,\mathscr{N}_u^{\mathsf{sing}}) & \xrightarrow{\delta} & \\ & \longrightarrow & \mathsf{H}^1(S,TS) & \longrightarrow & \mathsf{H}^1_D(S,E) & \longrightarrow & \mathsf{H}^1_D(S,N_u) & \longrightarrow & 0. \end{array}$$

2. The total moduli space of pseudoholomorphic curves

2.1. **Transversality.** Any deformation of a given J-holomorphic map $u:S\to X$ defines a path in the space $\mathscr{P}$ of pseudoholomorphic maps. Thus to construct such a deformation we want to equip the space $\mathscr{P}$ with a structure of a smooth Banach manifold.

Note that by definition the set $\mathscr{P}$ is the zero set of the section $\sigma_{\bar\partial}$ of the bundle $\mathscr{E}'$, *i.e.* the intersection of the images of $\sigma_{\bar\partial}$ and the zero-section σ_0. Thus we are interested in which points these sections meet transversally. The analysis of the problem leads to the following

Definition 2.1.1. Let $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{Z}$ be Banach manifolds with C^ℓ-smooth maps $f : \mathscr{Y} \to \mathscr{X}$ and $g : \mathscr{Z} \to \mathscr{X}$, $\ell \geqslant 1$. Define the *fiber product* $\mathscr{Y} \times_{\mathscr{X}} \mathscr{Z}$ by setting $\mathscr{Y} \times_{\mathscr{X}} \mathscr{Z} := \{(y,z) \in \mathscr{Y} \times \mathscr{Z} \ : \ f(y) = g(z)\}$. The map f is called *transversal to* g at a point $(y,z) \in \mathscr{Y} \times_{\mathscr{X}} \mathscr{Z}$ with $x := f(y) = g(z)$, and (y,z) is called a *transversality point*, if the map $df_y \oplus dg_z : T_y\mathscr{Y} \oplus T_z\mathscr{Z} \to T_x\mathscr{X}$ is *surjective* and admits a *closed complement* to its kernel. The set of transversality points $(y,z) \in \mathscr{Y} \times_{\mathscr{X}} \mathscr{Z}$ will be denoted by $\mathscr{Y} \times^{\pitchfork}_{\mathscr{X}} \mathscr{Z}$, with $\pitchfork$ symbolizing the transversality condition.

We say that $f : \mathscr{Y} \to \mathscr{X}$ is *transversal* to $g : \mathscr{Z} \to \mathscr{X}$ if every point $(y,z) \in \mathscr{Y} \times_{\mathscr{X}} \mathscr{Z}$ is a transversality point. In particular, if $\mathscr{Y}$ consists of a point $x \in \mathscr{X}$ and the imbedding $\{x\} \hookrightarrow \mathscr{X}$ is transversal to g, we call x a *regular value of* g. Note that by this definition any $x \in \mathscr{X} \backslash g(\mathscr{Z})$ is a regular value of g.

In the special case when the map $g : \mathscr{Z} \to \mathscr{X}$ is a closed imbedding, the fiber product $\mathscr{Y} \times_{\mathscr{X}} \mathscr{Z}$ is simply the preimage $f^{-1}\mathscr{Z}$ of $\mathscr{Z} \subset \mathscr{X}$. In particular, every point $(y,z) \in \mathscr{Y} \times_{\mathscr{X}} \mathscr{Z}$ is completely defined by its component $y \in \mathscr{Y}$, $z = f(y) \in \mathscr{Z} \subset \mathscr{X}$. In this case we simply say that $f : \mathscr{Y} \to \mathscr{X}$ is *transversal to* $\mathscr{Z}$ *at* $y \in \mathscr{Y}$, if and only if $(y, f(y))$ is a transversal point of $\mathscr{Y} \times_{\mathscr{X}} \mathscr{Z} \cong f^{-1}\mathscr{Z}$.

Lemma 2.1.1. *The set $\mathscr{Y} \times^{\pitchfork}_{\mathscr{X}} \mathscr{Z}$ is open in $\mathscr{Y} \times_{\mathscr{X}} \mathscr{Z}$ and is a C^ℓ-smooth Banach manifold with tangent space*

$$T_{(y,z)}\mathscr{Y} \times^{\pitchfork}_{\mathscr{X}} \mathscr{Z} = \mathsf{Ker}\big(df_y \oplus d(-g_z) : T_y\mathscr{Y} \oplus T_z\mathscr{Z} \to T_x\mathscr{X}\big). \tag{2.1.1}$$

Proof. Fix $w_0 := (y_0, z_0) \in \mathscr{Y} \times^{\pitchfork}_{\mathscr{X}} \mathscr{Z}$ and set $K_0 := \mathsf{Ker}\big(df_{y_0} \oplus dg_{z_0} : T_{y_0}\mathscr{Y} \oplus T_{z_0}\mathscr{Z} \to T_x\mathscr{X}\big)$. Let Q_0 be a closed complement to K_0. Then the map $df_{y_0} \oplus dg_{z_0} : Q_0 \to T_x\mathscr{X}$ is an isomorphism.

Due to the choice of Q_0, there exists a neighborhood $V \subset \mathscr{Y} \times \mathscr{Z}$ of (y_0, z_0) and C^ℓ-smooth maps $w' : V \to K_0$ and $w'' : V \to Q_0$, such that dw'_{w_0} (resp. dw''_{w_0}) is the projection from $T_{y_0}\mathscr{Y} \oplus T_{z_0}\mathscr{Z}$ onto K_0 (resp. onto Q_0), so that (w', w'') are coordinates in some smaller neighborhood $V_1 \subset \mathscr{Y} \times \mathscr{Z}$ of $w_0 = (y_0, z_0)$. It remains to consider the equation $f(y) = g(z)$ in new coordinates (w', w'') and apply the implicit function theorem. $\square$

Due to *Lemma 2.1.1*, the set $\mathscr{P}$ is a Banach manifold at those points $(u, J_S, J) \in \mathscr{P}$ where $\sigma_{\bar\partial}$ is transversal to σ_0. However, at any point $(u, J_S, J; 0)$ on the zero section σ_0 of $\mathscr{E}'$ we have the natural decomposition

$$T_{(u,J_S,J;0)}\mathscr{E}' = d\sigma_0\big(T_{(u,J_S,J)}(\mathscr{S} \times \mathscr{J}_S \times \mathscr{J})\big) \oplus \mathscr{E}'_{(u,J_S,J)}, \tag{2.1.2}$$

where the first component is the tangent space to the zero section of $\mathscr{E}'$ and the second one is the tangent space to the fiber $\mathscr{E}'_{(u,J_S,J)}$.

Let p_2 denote the projection on the second component. Then the transversality $\sigma_{\bar\partial}$ and σ_0 is equivalent to the surjectivity of the map $p_2 \circ d\sigma_{\bar\partial} : T_{(u,J_S,J)}(\mathscr{S} \times \mathscr{J}_S \times \mathscr{J}) \to \mathscr{E}'_{(u,J_S,J)}$, *i.e.* to the surjectivity of the operator

$$\begin{aligned}
&\nabla\sigma_{\bar\partial} : T_u L^{1,p}(S, X) \oplus T_{J_S}\mathbb{T}_g \oplus T_J\mathscr{J} \longrightarrow \mathscr{E}'_{(u,J_S,J)} \\
&\nabla\sigma_{\bar\partial} : (v, \dot J_S, \dot J) \longmapsto D_{(u,J)}v + J \circ du \circ \dot J_S + \dot J \circ du \circ J_S.
\end{aligned}$$

By *Definition 1.5.2*, the quotient of $\mathscr{E}'_{(u,J_S,J)}$ with respect to the image of $D_{u,J}$ is $\mathsf{H}^1_D(S, E_u)$. The induced map

$$T_{J_S}\mathscr{J}_S \ni \dot J_S \mapsto J \circ du \circ \dot J_S \in \mathsf{H}^1_D(S, E_u) \tag{2.1.3}$$

is also easy to describe. Recall that for a given complex structure J_S on S one has the Dolbeault isomorphism

$$\mathsf{H}^1(S,TS) = \mathsf{Coker}\left(\bar\partial : C^{2,\alpha}(S,TS) \longrightarrow C^{1,\alpha}_{(0,1)}(S,TS)\right) \tag{2.1.4}$$

with the operator $\bar\partial$ associated to J_S. Recall also that $C^{1,\alpha}_{(0,1)}(S,TS)$ is the tangent space $T_{J_S}\mathscr{J}_S$. This shows that the map (2.1.3) is the same as the map $J\circ du : \mathsf{H}^1(S,TS) \to \mathsf{H}^1_D(S,E_u)$ and, due to identity $J\circ du = du\circ J_S$ and *Corollary 1.5.4*, its cokernel is $\mathsf{H}^1_D(S,N_u)$.

It remains to study the image of $T_J\mathscr{J}$ in $\mathsf{H}^1_D(S,N_u)$.

Definition 2.1.2. For $(u,J_S,J)\in\mathscr{P}$ we define $\Psi=\Psi_{(u,J)} : T_J\mathscr{J} \to \mathscr{E}'_{(u,J_S,J)}$ by setting $\Psi_{(u,J)}(\dot J) := \dot J\circ du\circ J_S$. Let $\overline\Psi = \overline\Psi_{(u,J)} : T_J\mathscr{J} \to \mathsf{H}^1_D(S,N_u)$ be induced by Ψ. Finally, define $\mathscr{P}^* := \{(u,J_S,J)\in\mathscr{P} : u \text{ is injective in generic } z\in S\}$.

Remark. One can show that $\mathscr{P}\backslash\mathscr{P}^*$ consists of *multiple curves* for which the map $u:(S,J_S)\to(X,J)$ admits a factorization $u=u'\circ g$ for some non-trivial holomorphic branched covering $g:(S,J_S)\to(S',J'_S)$ and a J-holomorphic map $u':(S',J'_S)\to X$. On the other hand, for any $(u,J_S,J)\in\mathscr{P}^*$ the map u is a smooth imbedding outside finitely many points in S. For details see [Mi-Wh] or [Iv-Sh-1].

Lemma 2.1.2. *(Infinitesimal transversality). Let $(u,J_S,J)\in\mathscr{P}^*$. Then the operator $\overline\Psi : T_J\mathscr{J} \to \mathsf{H}^1_D(S,N_u)$ is surjective.*

Proof. Choose some nonempty open set $V\subset S$, such that $u|_V$ is an imbedding. Use Serre duality (*Lemma 1.5.1*) and find a basis $\psi_1,\dots\psi_l\in\mathsf{H}^0_D(S,N^*\otimes K_S)\cong\mathsf{H}^1_D(S,N)^*$.

Note that ψ_i satisfy the equation $D\psi_i=0$, where the operator $D=D_{N^*\otimes K_S}$ is of the form $\bar\partial+R$. One can show (see *e.g.* [Iv-Sh-1] or [H-L-S]) that any solution v of the equation $(\bar\partial+R)v=0$ is $L^{1,p}$-smooth and furthermore such a v is either identically zero or has isolated zeros.

This implies that there exist $I_1,\dots I_l\in C^\ell(S,N\otimes\Lambda^{(0,1)})$ with supports $\mathsf{supp}\,(I_i)$ in V such that the matrix $\left(\mathsf{Re}\int_S\psi_i\circ I_j\right)_{i,j=1}^l$ is non-degenerate. Since $u|_V$ is a $C^{\ell+1}$-smooth imbedding, any such I_i can be represented in the form $I_i=\dot J_i\circ du\circ J_S$ with some $\dot J_i\in C^\ell(X,\mathsf{End}(TX))$ with $J\circ\dot J_i+\dot J_i\circ J=0$. The latter relation means that $\dot J_i\in T_J\mathscr{J}$. □

Corollary 2.1.3. *$\mathscr{P}^*$ is a C^ℓ-smooth Banach manifold with the tangent space*

$$T_{(u,J_S,J)}\mathscr{P}^* = \left\{(v,\dot J_S,\dot J) \;:\; D_{u,J}v + J\circ du\circ\dot J_S + \dot J\circ du\circ J_S = 0\,\right\}. \tag{2.1.5}$$

Remark. $\mathscr{P}^*$ is in general smaller than the set $\mathscr{P}^\pitchfork := \sigma_{\bar\partial}\times^\pitchfork_{\mathscr{E}'}\sigma_0$ of all transversality points of $\mathscr{P}$. On the other hand, it is sufficient for applications to consider only the space $\mathscr{P}^*$.

2.2. **Moduli space of pseudoholomorphic curves.** The space $\mathscr{P}$ (see (1.3.7)) of all pseudoholomorphic maps is too big. Indeed, one has a natural right action of the group $\mathscr{D}\!\mathit{iff}_+(S)$ of the orientation preserving $C^{2,\alpha}$-smooth diffeomorphisms of S on the product $\mathscr{S}\times\mathscr{J}_S\times\mathscr{J}$ given by formula

$$(u,J_S,J)\in\mathscr{S}\times\mathscr{J}_S\times\mathscr{J}\,, g\in\mathscr{D}\!\mathit{iff}_+(S) \longrightarrow (u,J_S,J)\cdot g := (u\circ g, J_S\circ g, J), \tag{2.2.1}$$

such that $\mathscr{P}$ and $\mathscr{P}^*$ are invariant with respect to this action. It is natural to consider $(u,J_S,J)\in\mathscr{P}$ and $(u,J_S,J)\cdot g$ as two parameterization of the same a J-curve. In other words, we are interested in the quotient space $\mathscr{P}/\mathscr{D}\!\mathit{iff}_+(S)$ rather than the space $\mathscr{P}$

itself. As in Yang-Mills theory one can treat the group $\mathscr{D}\!\mathit{iff}_+(S)$ as the gauge group of the problem and the quotient as the corresponding moduli space. Again as in Yang-Mills theory it is useful to know in which points the quotient spaces $\mathscr{P}/\mathscr{D}\!\mathit{iff}_+(S)$ is a Banach manifold.

First we consider the action of the group $\mathscr{D}\!\mathit{iff}_+(S)$ on the space $\mathscr{J}_S$. Note that if $J'_S, J''_S \in \mathscr{J}_S$ are related by $J''_S = J'_S \circ g$ for some C^1-diffeomorphism $g : S \to S$, then g is (J'_S, J''_S)-holomorphic. Since J'_S and J''_S are $C^{1,\alpha}$-smooth, elliptic regularity implies that g is $C^{2,\alpha}$-smooth.

Further, it is known that the action of $\mathscr{D}\!\mathit{iff}_+(S)$ on $\mathscr{J}_S$ admits a global finite-dimensional slice. To describe this slice we recall some standard facts from Teichmüller theory.

Denote by $\mathbb{T}_g$ the Teichmüller space of marked complex structures on S. This is a complex manifold of dimension

$$\mathsf{dim}_{\mathbb{C}}\mathbb{T}_g = \begin{cases} 0 & \text{if } g=0; \\ 1 & \text{if } g=1; \\ 3g-3 & \text{if } g \geqslant 2; \end{cases} \tag{2.2.2}$$

which can be completely characterized in the following way.

Proposition 2.2.1. *The product $S \times \mathbb{T}_g$ admits a (non-unique) complex (i.e. holomorphic) structure $J_{S\times\mathbb{T}}$ such that:*

i) The natural projection $\pi_{\mathbb{T}} : S \times \mathbb{T}_g \to \mathbb{T}_g$ is holomorphic, so that for any $\tau \in \mathbb{T}_g$ the identification $S \cong S \times \{\tau\}$ induces the complex structure $J_S(\tau) := J_{S\times\mathbb{T}}|_{S\times\{\tau\}}$ on S;

ii) For any complex structure I_S on S there exist a uniquely defined $\tau \in \mathbb{T}_g$ and a diffeomorphism $f : S \to S$ homotopic to the identity map $\mathsf{Id}_S : S \to S$ such that $I_S = f^* J_S(\tau)$;

iii) Moreover, for any finite-dimensional manifold Y and any smooth map $H : Y \to \mathscr{J}_S$ there exist maps $F : Y \to \mathscr{D}\!\mathit{iff}_+(S)$ and $h : Y \to \mathbb{T}$ such that $H(y) = (F(y))^* h(y)$;

iv) The group $\mathbf{G}$ of automorphisms of $S \times \mathbb{T}_g$ preserving the projection onto $\mathbb{T}_g$ is

$$\mathbf{G} = \begin{cases} \mathbf{PGl}(2,\mathbb{C}) & \text{for } g=0, \\ \mathbf{Sp}(2,\ \mathbb{Z}) \ltimes T^2 & \text{for } g=1, \\ \text{discrete} & \text{for } g \geqslant 2; \end{cases} \tag{2.2.3}$$

v) The tangent space to $\mathbb{T}_g$ at τ is canonically isomorphic to $\mathsf{H}^1(S, TS)$ where S is equipped with the structure $J_S(\tau)$. The group $\mathsf{H}^0(S, TS)$ is canonically isomorphic to the Lie algebra of $\mathbf{G}$.

We shall assume that such a structure $J_{S\times\mathbb{T}}$ is fixed. Then we obtain an imbedding $\mathbb{T} \hookrightarrow \mathscr{J}_S$ given by $\tau \in \mathbb{T} \mapsto J_S(\tau) \in \mathscr{J}_S$. Using it, we identify $\mathbb{T}$ with its image in $\mathscr{J}_S$. For any $J_S \in \mathbb{T}_g$ this induces a monomorphism $T_{J_S}\mathbb{T}_g \hookrightarrow T_{J_S}\mathscr{J}_S = C^{1,\alpha}_{(0,1)}(S, TS)$. Now, the isomorphism $T_{J_S}\mathbb{T}_g \cong \mathsf{H}^1(S, TS)$ mentioned in *v)* is obtained as the composition

$$T_{J_S}\mathbb{T}_g \hookrightarrow T_{J_S}\mathscr{J}_S = C^{1,\alpha}_{(0,1)}(S,TS) \longrightarrow C^{1,\alpha}_{(0,1)}(S,TS)/\bar\partial\big(C^{2,\alpha}(S,TS)\big) = \mathsf{H}^1(S,TS). \tag{2.2.4}$$

By our construction, any $\mathscr{D}\!\mathit{iff}_+(S)$-orbit in $\mathscr{J}_S$ intersects $\mathbb{T}$. This implies that instead of $\mathscr{P}/\mathscr{D}\!\mathit{iff}_+(S)$ we can consider the quotient $\mathscr{P} \cap (\mathscr{S} \times \mathbb{T} \times \mathscr{J}_S)$ by the action of $\mathbf{G}$.

Definition 2.2.1. Let $\widehat{\mathscr{M}} := \mathscr{P}^* \cap (\mathscr{S} \times \mathbb{T} \times \mathscr{J}_S)$ and use the same the notations to the restrictions of the bundles $\mathscr{E}$ and $\mathscr{E}'$ onto $\widehat{\mathscr{M}}$ and for the induced operator $D : \mathscr{E} \to \mathscr{E}'$. The quotient $\mathscr{M} := \widehat{\mathscr{M}}/\mathbf{G}$ is the *total moduli space of parameterized pseudoholomorphic*

curves. It is equipped with the projection $\mathsf{pr}_{\mathscr{J}} : \mathscr{M} \to \mathscr{J}$. Elements of $\mathscr{M}$ are denoted by $[u, J]$. To indicate the surface S, the ambient manifold X, and the homology class $[C] \in \mathsf{H}_2(X, \mathbb{Z})$ involved in the definition of $\mathscr{M}$ we shall also use the notation $\mathscr{M}(S, X, [C])$. The same meaning have the notations $\widehat{\mathscr{M}}(S, X, [C])$ and $\mathscr{P}(S, X, [C])$.

Lemma 2.2.2. . *i) The projection $\widehat{\mathsf{pr}} : \widehat{\mathscr{M}} \longrightarrow \mathscr{M}$ is a principal $\mathbf{G}$-bundle.*

ii) The bundles $\mathscr{E}$ and $\mathscr{E}'$ over $\widehat{\mathscr{M}}$ admit a natural C^ℓ-smooth $\mathbf{G}$-action such that $D : \mathscr{E} \to \mathscr{E}'$ is $\mathbf{G}$-invariant.

iii) For any $J \in \mathscr{J}$ and any non-multiple J-holomorphic map $u : S \to X$ there exists a diffeomorphism $\varphi : S \to S$ such that $[u \circ \varphi, J]$ lies in $\mathscr{M}$. Moreover, such element of $\mathscr{M}$ is unique.

Remarks. 1. Part *ii)* of the lemma is equivalent to the existence of C^ℓ-smooth bundles $\mathscr{E}_{\mathscr{M}}$ and $\mathscr{E}'_{\mathscr{M}}$ over $\mathscr{M}$ and a $C^{\ell-1}$-smooth bundle homomorphism $D_{\mathscr{M}} : \mathscr{E}_{\mathscr{M}} \to \mathscr{E}'_{\mathscr{M}}$ which lift to the corresponding objects over $\widehat{\mathscr{M}}$. Later on we drop the sub-index $_{\mathscr{M}}$, so that, *e.g.* $\mathscr{E}$ will denote also the corresponding bundle over $\mathscr{M}$.

2. Our main interest is the space $\mathscr{M}$. However, in the proofs below we shall mostly work with $\widehat{\mathscr{M}}$. The reason is that an element $(u, J_S, J) \in \widehat{\mathscr{M}}$ fixes a parameterization of a pseudoholomorphic curve, whereas $[u, J] \in \mathscr{M}$ defines only an appropriate equivalence class of parameterizations.

Proof. Part *i)*.

Case $g \geqslant 2$. It is known that in this case $\mathbf{G}$ acts properly discontinuously on $\mathbb{T}_g$. This implies that the same is true for the action of $\mathbf{G}$ on $\widehat{\mathscr{M}}$. Moreover, it is clear that $\mathbf{G}$ acts freely on $\widehat{\mathscr{M}}$. Consequently, the map $\widehat{\mathscr{M}} \longrightarrow \mathscr{M} = \widehat{\mathscr{M}}/\mathbf{G}$ is simply an (unbranched) covering.

Case $g = 0$. In this case $S = S^2$, $\mathbb{T}_0 = \{J_{\mathsf{st}}\}$, and the action of $\mathbf{G}$ on S is generated by holomorphic vector fields, *i.e.* by the space $\mathsf{H}^0(S, TS)$. One can show that the action of $\mathbf{G}$ on $\widehat{\mathscr{M}}$ is generated by vector fields

$$(u, J_{\mathsf{st}}, J) \in \widehat{\mathscr{M}} \mapsto (du(v), 0, 0) \in T_{(u, J_{\mathsf{st}}, J)}\widehat{\mathscr{M}} \quad \text{with } v \in \mathsf{H}^0(S, TS) \text{ fixed.} \tag{2.2.5}$$

In particular, the action is *smooth* or, more precisely, C^ℓ-smooth.

Consequently, for a given $(u^0, J_{\mathsf{st}}, J^0) \in \widehat{\mathscr{M}}$ we can find a closed complementing space $\mathscr{V} \subset T_{(u^0, J_{\mathsf{st}}, J^0)}\widehat{\mathscr{M}}$ to $(u^0(\mathsf{H}^0(S, TS)), 0, 0)$. Represent it as the tangent space of a submanifold $\mathscr{W} \subset \widehat{\mathscr{M}}$ through $(u^0, J_{\mathsf{st}}, J^0)$, $T_{(u^0, J_{\mathsf{st}}, J^0)}\mathscr{W} = \mathscr{V}$. If $\mathscr{V}$ is chosen sufficiently small, then it intersects every orbit $\mathbf{G} \cdot (u, J_{\mathsf{st}}, J)$ transversally in exactly one point. Moreover, we have a $\mathbf{G}$-invariant diffeomorphism $\mathbf{G} \cdot \mathscr{W} \cong \mathbf{G} \times \mathscr{W}$, so that $\mathscr{W}$ is a local slice of $\mathbf{G}$-action at $(u^0, J_{\mathsf{st}}, J^0)$. This equips the quotient $\widehat{\mathscr{M}}/\mathbf{G}$ with a structure of a smooth Banach manifold such that the projection $\mathscr{W} \to \mathscr{M} = \widehat{\mathscr{M}}/\mathbf{G}$ is a C^ℓ-smooth chart.

Case $g = 1$ is a combination of the above two cases. First we consider the action of $T^2 \triangleleft \mathbf{G}$. The existence of a local T^2-slice $\mathscr{W}$ through any given $(u^0, J^0_S, J^0) \in \widehat{\mathscr{M}}$ can be shown by copying the construction of *Case* $g = 0$. This implies that $\widehat{\mathscr{M}} \longrightarrow \widehat{\mathscr{M}}/T^2$ is a principle T^2-bundle. Then we repeat the argument of *Case* $g \geqslant 2$ and show that $\widehat{\mathscr{M}}/T^2 \to \widehat{\mathscr{M}}/\mathbf{G}$ is an unbranched covering with the group $\mathbf{Sp}(2, \mathbb{Z}) = \mathbf{G}/T^2$. This completes the proof of part *i)*.

Part ii). The action of $\mathbf{G}$ extends in a natural way to an action on $\mathscr{X} := S \times \mathscr{S} \times \mathbb{T} \times \mathscr{J}$. The evaluation map $\mathsf{ev} : \mathscr{X} \to X$, $\mathsf{ev}(z, u, J_S, J) := u(z)$ is $\mathbf{G}$-equivariant. Consequently,

the bundle $E := \mathsf{ev}^*TX$ over $\mathscr{X}$ is equipped with the natural $\mathbf{G}$-action. The action of $\mathbf{G}$ on E induces the actions on section spaces $\mathscr{E}$ and $\mathscr{E}'$. Since all constructions are natural, $D : \mathscr{E} \to \mathscr{E}'$ is $\mathbf{G}$-invariant.

Finally, it remains to note that the action of $\mathbf{G}$ on the bundles $\mathscr{E}$ and $\mathscr{E}'$ over $\widehat{\mathscr{M}}$ is C^ℓ-smooth.

Part iii) of the lemma states the universality property of $\mathscr{M}$ which easily follows from the definitions. □

Corollary 2.2.3. *$\mathscr{M}$ is a C^ℓ-smooth Banach manifold and $\pi_{\mathscr{J}} : \mathscr{M} \to \mathscr{J}$ is a Fredholm map. For $[u, J] \in \mathscr{M}$ there exist natural isomorphisms*

$$\mathsf{Ker}\,(d\pi_{\mathscr{J}} : T_{[u,J]}\mathscr{M} \to T_J\mathscr{J}) \;\cong\; \mathsf{H}^0_D(S, \mathscr{N}_u),$$
$$\mathsf{Coker}\,(d\pi_{\mathscr{J}} : T_{[u,J]}\mathscr{M} \to T_J\mathscr{J}) \;\cong\; \mathsf{H}^1_D(S, \mathscr{N}_u).$$

In particular, the index of $\pi_{\mathscr{J}}$ is equal to

$$\mathsf{ind}_{\mathbb{R}}\pi_{\mathscr{J}} = \chi_{\mathbb{R}}(\mathscr{N}_u) = 2\big(\mu + (n-3)(1-g)\big), \tag{2.2.6}$$

where $\mu = \langle c_1(X), [C]\rangle$.

Proof. The C^ℓ-smooth structure on $\mathscr{M}$ is is the quotient structure defined by the the C^ℓ-smooth $\mathbf{G}$-action on $\widehat{\mathscr{M}}$.

Using *Corollary 2.1.3* we see that the tangent space to $\widehat{\mathscr{M}}$ is

$$T_{(u,J_S,J)}\widehat{\mathscr{M}} = \big\{(v, \dot{J}_S, \dot{J}) : \dot{J}_S \in T_{J_S}\mathbb{T},\ D_{u,J}v + J\circ du \circ \dot{J}_S + \dot{J}\circ du\circ J_S = 0\big\}. \tag{2.2.7}$$

Consider the natural projection $\pi_{\mathscr{P}} : \mathscr{P}^* \to \mathscr{J}$, $(u, J_S, J) \mapsto J$ with the differential $d\pi_{\mathscr{P}} : T_{(u,J_S,J)}\mathscr{P}^* \to T_J\mathscr{J}$ given by $(v, \dot{J}_S, \dot{J}) \in T_{(u,J_S,J)}\mathscr{P}^* \mapsto \dot{J} \in T_J\mathscr{J}$.

The kernel $\mathsf{Ker}\,(d\pi_{\mathscr{P}})$ consists of solutions $v \in \mathscr{E}_{(u,J)}$ of the equation

$$D_{u,J}v + J\circ du\circ \dot{J}_S = 0 \tag{2.2.8}$$

with $\dot{J}_S \in T_{J_S}\mathbb{T}$. Since the map $\widehat{\pi} : \widehat{\mathscr{M}} \to \mathscr{M}$ is a principle $\mathbf{G}$-bundle, the kernel $\mathsf{Ker}\,(d\pi_{\mathscr{J}} : T_{(M,J)}\mathscr{M} \to T_J\mathscr{J})$ is obtained from $\mathsf{Ker}\,(d\pi)$ by taking the quotient by the tangent space to the fiber $\mathbf{G}\cdot(u, J_S, J)$ which is equal to $du(\mathsf{H}^0(S,TS))$. Using the relations $\mathsf{H}^0(S,TS) = \mathsf{Ker}\,(\bar\partial_{TS} : L^{1,p}(S,TS) \to L^p(S, TS\otimes\Lambda^{(0,1)}S)$, $T_{J_S}\mathbb{T}_g \cong \mathsf{H}^1(S,TS) = \mathsf{Coker}\,(\bar\partial_{TS})$, and $du\circ\bar\partial_{TS} = D_{(u,J)}\circ du$, we conclude that the space $\mathsf{Ker}\,(d\pi_{\mathscr{J}})$ is isomorphic to the quotient

$$\{v \in L^{1,p}(S, E_u) : Dv = du(\varphi) \text{ for some } \varphi \in L^p(S, TS\otimes\Lambda^{(0,1)}S)\}/du\big(L^{1,p}(S,TS)\big). \tag{2.2.9}$$

Hence, by *Lemma 1.5.3*, $\mathsf{Ker}\,(d\pi_{\mathscr{J}} : T_{[u,J]}\mathscr{M} \to T_J\mathscr{J}) \cong \mathsf{H}^0_D(S,\mathscr{N}_u)$. In particular, $\mathsf{Ker}\,(d\pi_{\mathscr{J}})$ is finite dimensional.

Similarly, the image of $d\pi_{\mathscr{J}}$ consists of those $\dot{J}$ for which the equation

$$D_{u,J}v + J\circ du\circ\dot{J}_S + \dot{J}\circ du\circ J_S = 0 \tag{2.2.10}$$

has a solution $(v, \dot{J}_S)$. Using (1.5.8) and *Lemma 1.5.3* we obtain the relations $\mathsf{Im}\,(d\pi_{\mathscr{J}}) = \mathsf{Ker}\,\overline{\Psi}$ and $\mathsf{Coker}\,(d\pi) \cong \mathsf{H}^1_D(S, N_u)$.

This implies the Fredholm property for the projection $\pi_{\mathscr{J}} : \mathscr{M} \to \mathscr{J}$ and the formula $\mathsf{ind}(d\pi_{\mathscr{J}}) = \mathsf{ind}(\mathscr{N}_u)$. □

We conclude the paragraph with a description of the deformations of non-closed pseudoholomorphic curves.

Definition 2.2.2. Let $\overline{S} = S \cup \partial S$ be a compact *non-closed* oriented surface with the boundary ∂S consisting of finitely many circles. Denote by $\mathscr{J}_S$ the (Banach) space of complex structure on S which are compatible with the orientation of S and C^ℓ-smooth up to boundary ∂S. As usual let

$$\mathscr{P}(S,X) := \{(u, J_S, J) \in L^{1,p}(S,X) \times \mathscr{J}_S \times \mathscr{J} : \bar\partial_{J_S,J} u = 0\}, \tag{2.2.11}$$

the space of pseudoholomorphic maps equipped it with the natural projections $\mathsf{pr}_{\mathscr{J}_S} : \mathscr{P}(S,X) \to \mathscr{J}_S$ and $\mathsf{pr}_{\mathscr{J}} : \mathscr{P}(S,X) \to \mathscr{J}$. The fibers of the projections are denoted by $\mathscr{P}(S,X,J) = \mathsf{pr}_{\mathscr{J}}^{-1}(J)$, $\mathscr{P}(S,J_S,X) = \mathsf{pr}_{\mathscr{J}_S}^{-1}(J_S)$, and $\mathscr{P}(S,J_S,X,J) = \mathscr{P}(S,J_S,X) \cap \mathscr{P}(S,X,J)$ respectively.

Lemma 2.2.4. *i) Let S be a non-closed oriented surface. Then*
i) the space $\mathscr{P}(S,X)$ is a Banach submanifolds of $L^{1,p}(S,X) \times \mathscr{J}_S \times \mathscr{J}$;
ii) For any $(u, J_S, J) \in \mathscr{P}(S,X)$, the operators $d\mathsf{pr}_{\mathscr{J}} : T_{(u,J_S,J)}\mathscr{P}(S,X) \to T_J\mathscr{J}$ and $d\mathsf{pr}_{\mathscr{J}_S} : T_{(u,J_S,J)}\mathscr{P}(S,X) \to T_{J_S}\mathscr{J}$ are surjective and split. In particular, $\mathscr{P}(S;X,J)$ and $\mathscr{P}(S,J_S;X)$ are are Banach submanifolds of $\mathscr{P}(S,X)$;
iii) For any $(u, J_S, J) \in \mathscr{P}(S,X)$, the submanifolds $\mathscr{P}(S;X,J)$ and $\mathscr{P}(S,J_S;X)$ are transversal in (u, J_S, J); in particular, $\mathscr{P}(S,J_S;X,J) = \mathscr{P}(S;X,J) \cap \mathscr{P}(S,J_S;X)$ is also a Banach submanifold;
iv) The the tangent spaces are given by

$$\begin{aligned} T_{(u,J_S,J)}\mathscr{P}(S,X) = \big\{ (v, \dot J_S, \dot J) \in T_u L^{1,p}(S,X) \times T_{J_S}\mathscr{J}_S \times T_J\mathscr{J} : \\ D_{u,J} v + \dot J \circ du \circ J_S + J \circ du \circ \dot J_S = 0 \big\}; \end{aligned} \tag{2.2.12}$$

$$T_{(u,J_S)}\mathscr{P}(S;X,J) = \big\{ (v, \dot J_S, \dot J) \in T_{(u,J_S,J)}\mathscr{P}(S,X) : \dot J = 0 \big\}; \tag{2.2.13}$$

$$T_{(u,J)}\mathscr{P}(S,J_S;X) = \big\{ (v, \dot J_S, \dot J) \in T_{(u,J_S,J)}\mathscr{P}(S,X) : \dot J_S = 0 \big\}; \tag{2.2.14}$$

$$T_u\mathscr{P}(S,J_S;X,J) = \big\{ (v, \dot J_S, \dot J) \in T_{(u,J_S,J)}\mathscr{P}(S,X) : \dot J_S = 0 = \dot J \big\}. \tag{2.2.15}$$

Proof. The lemma is obtained by the transversality techniques of *Paragraph 2.1* using the following claim: *For any $(u, J_S, J) \in \mathscr{P}(S,X)$ the operator $D_{u,J} : L^{1,p}(C, E_u) \to L^p_{(0,1)}(C, E_u)$ is surjective.* Since $D_{u,J}$ is elliptic, this is a standard fact following from compactness and non-closedness of S. □

2.3. **Transversality II.** Before stating further results, we introduce some new notation. Here S is a *closed* real surface.

Definition 2.3.1. Let Y be a C^ℓ-smooth finite-dimensional manifold, possibly with non-empty C^ℓ-smooth boundary ∂Y, and $h : Y \to \mathscr{J}$ a C^ℓ-smooth map. Define the *relative Moduli space*

$$\mathscr{M}_h := Y \times_{\mathscr{J}} \mathscr{M} \cong \{ (u, J_S, y) \in \mathscr{S} \times \mathbb{T}_g \times Y : (u, J_S, h(y)) \in \mathscr{P}^* \} / \mathbf{G} \tag{2.3.1}$$

with the natural projection $\pi_h : \mathscr{M}_h \to Y$. In the special case $Y = \{J\} \hookrightarrow \mathscr{J}$, we obtain the Moduli space of J-holomorphic curves $\mathscr{M}_J := \pi_{\mathscr{J}}^{-1}(J)$. The projection $\pi_h : \mathscr{M}_h \to Y$ is a fibration with a fiber $\pi_h^{-1}(y) = \mathscr{M}_{h(y)}$. We shall denote elements of $\mathscr{M}_h$ by $[u, y]$, where $u : S \to X$ is a $h(y)$-holomorphic map.

The next two lemmas follow from the transversality theory.

Lemma 2.3.1. *Let Y be a C^ℓ-smooth finite-dimensional manifold, and $h : Y \to \mathscr{J}$ a C^ℓ-smooth map. Then $\mathscr{M}_h$ is a C^ℓ-smooth manifold in some neighborhood of a point $[u,y] \in \mathscr{M}_h$ with $J := h(y)$ if and only if the map $\overline{\Psi}_{u,J} \circ dh : T_uY \to \mathsf{H}^1_D(S,N_u)$ is surjective. In this case the tangent space to $\mathscr{M}_h$ is*

$$T_{[u,y]}\mathscr{M}_h = \mathsf{Ker}\left(D \oplus \Psi \circ dh : \mathscr{E}_{u,h(y)} \oplus T_yY \longrightarrow \mathscr{E}'_{u,h(y)}\right) / \, du(\mathsf{H}^0(S,TS)) \tag{2.3.2}$$

Proof. We reformulate the transversality condition and use *Lemma 2.1.1*. □

Lemma 2.3.2. *i) There exists a Baire subset $\mathscr{J}^{\mathsf{reg}} \subset \mathscr{J}$ such that any $J \in \mathscr{J}$ is a regular value of $\pi_{\mathscr{J}} : \mathscr{M} \to \mathscr{J}$.*

ii) There exists a Baire subset $\mathscr{V}$ in the space $C^\ell([0,1],\ \mathscr{J})$, such that any map $h : [0,1] \to \mathscr{J}$ from $\mathscr{V}$ is transversal to $\pi_{\mathscr{J}} : \mathscr{M} \to \mathscr{J}$ and both $h(0)$ and $h(1)$ are regular values of $\pi_{\mathscr{J}}$.

Remark. In general, for any finite-dimensional manifold Y with boundary ∂Y there exists a Baire subset $\mathscr{V} \subset C^\ell(Y,\ \mathscr{J})$ such that any $h \in \mathscr{V}$, as well as its restriction $h|_{\partial Y}$ are transversal to $\pi_{\mathscr{J}}$. The proof uses the Sard lemma.

Lemma 2.3.3. *Suppose that S is the sphere S^2 and $\mathsf{dim}_{\mathbb{R}}(X) = 4$. Then the exists a connected Baire subset $\mathscr{J}^{\mathsf{reg}} \subset \mathscr{J}$ such that any $J \in \mathscr{J}$ is a regular value of $\pi_{\mathscr{J}} : \mathscr{M} \to \mathscr{J}$. Moreover, any $J_0, J_1 \in \mathscr{J}^{\mathsf{reg}}$ can be connected by a smooth path $h : [0,1] \to \mathscr{J}^{\mathsf{reg}}$.*

Proof. By *Lemma 2.3.2*, there exists a Baire subset $\mathscr{J}^{\mathsf{reg}} \subset \mathscr{J}$ such that any $J \in \mathscr{J}^{\mathsf{reg}}$ is a regular value of $\pi_{\mathscr{J}}$. Further, any $J_0, J_1 \in \mathscr{J}^{\mathsf{reg}}$ can be connected by a smooth path $h : [0,1] \to \mathscr{J}$, transversal to $\pi_{\mathscr{J}}$.

For any such path $h : [0,1] \to \mathscr{J}$ and any $[u,t] \in \mathscr{M}_h$ with $h(t) = J$ the map $\overline{\Psi}_{u,J} \circ dh : T_t[0,1] \cong \mathbb{R} \longrightarrow \mathsf{H}^1(S,N_u)$ is surjective by *Lemma 2.3.1*. Consequently, for such u and J we have $\mathsf{dim}_{\mathbb{R}}\mathsf{H}^1_D(S,\mathscr{N}_u) = \mathsf{dim}_{\mathbb{R}}\mathsf{H}^1_D(S,N_u) \leqslant 1$.

Recall that the difference $\mathsf{dim}_{\mathbb{R}}\mathsf{H}^0_D(S,N_u) - \mathsf{dim}_{\mathbb{R}}\mathsf{H}^1_D(S,N_u)$ is *even*, see (1.5.4). Hence, if $\mathsf{dim}_{\mathbb{R}}\mathsf{H}^1(S,N_u) = 1$ then $\mathsf{H}^1_D(S,N_u)$ should also be nontrivial. On the other hand, the condition $\mathsf{dim}_{\mathbb{R}}(X) = 4$ implies that N_u is a *line bundle*. But, in view of *Lemma 1.5.2*, on the sphere $S = S^2$ one of the spaces $\mathsf{H}^i_D(S,N_u)$ must be trivial.

Thus, we see that for any such path h and any $[u,t] \in \mathscr{M}_h$ one has $\mathsf{H}^1_D(S,\mathscr{N}_u) = 0$. This means that h takes values in $\mathscr{J}^{\mathsf{reg}}$. □

For higher genus $g(S) \geqslant 1$ we have a similar, but weaker result.

Lemma 2.3.4. *Assume that $\mathsf{dim}_{\mathbb{R}}X = 4$ and $g := g(S) \geqslant 1$ and set $\mu := c_1(X)[u(S)]$. Then*

$$\mu \leqslant \mathsf{dim}_{\mathbb{C}}\mathsf{H}^0(S,\mathscr{N}_u^{\mathsf{sing}}) \leqslant \mu + g - 1 \tag{2.3.3}$$

for any $[u,J] \in \mathscr{M}$ with $\mathsf{H}^1(S,N_u) \cong \mathbb{R}$.

Proof. The condition $\mathsf{dim}_{\mathbb{R}}X = 4$ means that N_u is a *line* bundle. Thus, by *Lemma 1.5.2*, $\mathsf{H}^1(S,N_u) \cong \mathbb{R}$ implies $c_1(N_u) \leqslant 2g-2$. Further, since $\mathsf{dim}_{\mathbb{R}}\mathsf{H}^1(S,N_u) = 1$ and $\mathsf{ind}_{\mathbb{R}}D^N_{u,J} = \mathsf{dim}_{\mathbb{R}}\mathsf{H}^0_D(S,N_u) - \mathsf{dim}_{\mathbb{R}}\mathsf{H}^1_D(S,N_u)$ is *even*, see (1.5.4), we conclude that $\mathsf{dim}_{\mathbb{R}}\mathsf{H}^0_D(S,N_u) \geqslant 1$. Consequently, $\mathsf{ind}_{\mathbb{R}}D^N_{u,J} \geqslant 0$. The formula (1.5.4) for $\mathsf{ind}_{\mathbb{R}}D^N_{u,J}$ yields $c_1(N_u) \geqslant g-1$. Finally, the definition of $\mathscr{N}_u$ yields the relation $\mu = c_1(X)[u(S)] = c_1(E_u) = c_1(TS) + c_1(N_u) + \mathsf{dim}_{\mathbb{C}}\mathsf{H}^0(S,\mathscr{N}_u^{\mathsf{sing}})$. □

2.4. **Pseudoholomorphic curves through fixed points.** In this paragraph we consider the total moduli space of pseudoholomorphic curves passing through given fixed points $x_1, \ldots, x_m \in X$. Gromov in [Gro] proposed a method to reduce this problem to the one of pseudoholomorphic curves without such constraints. The idea is to blow up X in the points $x_1, \ldots, x_m$ and consider the curves on the blown-up space $\tilde{X}$. He has shown that a C^ℓ-smooth almost complex structure J on X lifts to a $C^{\ell-1}$-smooth almost complex structure $\tilde{J}$ on $\tilde{X}$ such that the natural projection $\mathsf{pr} : \tilde{X} \to X$ is holomorphic and such that every J-holomorphic curve C in X passing through $x_1, \ldots, x_m$ lifts to a unique $\tilde{J}$-holomorphic curve $\tilde{C}$ in $\tilde{X}$ with $C = \mathsf{pr}(\tilde{C})$.

Our aim here is to make an explicit construction for the moduli space of pseudoholomorphic curves passing through fixed points. Since the construction is simply a modification of the case $m = 0$ where no points are marked, we shall mostly skip or merely indicate proof of claims.

We begin by introducing some notation. Denote by $\boldsymbol{x} = (x_1, \ldots x_m)$ the tuple of fixed points on X, which are supposed to be pairwise distinct. Also fix a tuple $\boldsymbol{z} = (z_1, \ldots, z_m)$ of pairwise distinct points on the surface S. Define

$$\mathscr{S}(\boldsymbol{z}, \boldsymbol{x}) := \{u \in \mathscr{S} = L^{1,p}(S, X) : u(z_i) = x_i\};$$

$$\mathscr{P}(S, \boldsymbol{z}; X, \boldsymbol{x}) := \{(u, J_S, J) \in \mathscr{P} : u \in \mathscr{S}(\boldsymbol{z}, \boldsymbol{x})\};$$

$$\mathscr{P}^*(S, \boldsymbol{z}; X, \boldsymbol{x}) := \mathscr{P}(S, \boldsymbol{z}; X, \boldsymbol{x}) \cap \mathscr{P}^*(S, X).$$

The linearization of the conditions $u(z_i) = x_i$ yields the equations $v(z_i) = 0$ for $v \in T_u\mathscr{S} = L^{1,p}(S, E_u)$. Denote as above $E = E_u := u^*TX$, and set $E_i = E_{u,i} := (E_u)_{z_i} = T_{u(z_i)}X$. Then we obtain the bundle $E_{\boldsymbol{z}}$ over $\mathscr{S}$ with a fiber $(E_{\boldsymbol{z}})_u := \oplus E_{u,i}$ equipped with the natural *evaluation homomorphism* $\mathsf{ev}_{\boldsymbol{z}} : \mathscr{E} \to E_{\boldsymbol{z}}$

$$\mathsf{ev}_{\boldsymbol{z}} : v \in \mathscr{E}_u = L^{1,p}(S, E) \mapsto (v(z_1), \ldots, v(z_m)) \in E_{\boldsymbol{z}}.$$

It is easy to see that $\mathsf{ev}_{\boldsymbol{z}} : \mathscr{E} \to E_{\boldsymbol{z}}$ is surjective. This means that the equations $u(z_i) = x_i$ are transversal and implies that $\mathscr{S}(\boldsymbol{z}, \boldsymbol{x})$ is a Banach submanifold of $\mathscr{S}$ with the tangent space

$$T_u\mathscr{S}(\boldsymbol{z}, \boldsymbol{x}) = \{v \in T_u\mathscr{S} = L^{1,p}(S, E_u) : v(z_i) = 0\}. \tag{2.4.1}$$

The same argument shows that $\mathscr{P}^*(S, \boldsymbol{z}; X, \boldsymbol{x})$ is also a Banach submanifold of $\mathscr{P}^*(S, X)$ with the tangent space

$$T_u\mathscr{P}^*(S, \boldsymbol{z}; X, \boldsymbol{x}) = \{(v, \dot{J}_S, \dot{J}) \in T_u\mathscr{P}^*(S, X) : v(z_i) = 0\}.$$

Let $\mathscr{D}\!\mathit{iff}_+(S, \boldsymbol{z})$ be the subgroup of those $g \in \mathscr{D}\!\mathit{iff}_+(S)$ which fix the marked points $z_1, \ldots, z_m$. Then $\mathscr{D}\!\mathit{iff}_+(S, \boldsymbol{z})$ leaves the subsets $\mathscr{S}(\boldsymbol{z}, \boldsymbol{x}) \subset \mathscr{S}$ and $\mathscr{P}^*(S, \boldsymbol{z}; X, \boldsymbol{x}) \subset \mathscr{P}(S, X)$ invariant. So we can define the *total moduli space of pseudoholomorphic curves through the given points* $x_1, \ldots, x_m$ as the quotient $\mathscr{M}(\boldsymbol{x}) := \mathscr{P}^*(S, \boldsymbol{z}; X, \boldsymbol{x})/\mathscr{D}\!\mathit{iff}_+(S, \boldsymbol{z})$. This space is equipped with the natural projection $\pi_{\mathscr{J}} : \mathscr{M}(\boldsymbol{x}) \to \mathscr{J}$ defined in an obvious way.

The smooth structure on $\mathscr{M}(\boldsymbol{x})$ is constructed in the same way as it was for $\mathscr{M}$. First, one constructs a global slice on the action of $\mathscr{D}\!\mathit{iff}_+(S, \boldsymbol{z})$ on $\mathscr{J}_S$. To do this, we consider the action of the component of $\mathscr{D}\!\mathit{iff}_0(S, \boldsymbol{z})$ the group $\mathscr{D}\!\mathit{iff}_+(S, \boldsymbol{z})$ containing the identity. The quotient $\mathscr{J}_S/\mathscr{D}\!\mathit{iff}_0(S, \boldsymbol{z})$ is the *Teichmüller space* $\mathbb{T}_{g,m}$ *of complex structures on a Riemann surface of genus* $g = g(S)$ *with* m *punctures*. The marked points $z_1, \ldots, z_m$ are the positions of punctures.

As in the case $m = 0$, one can imbed $\mathbb{T}_{g,m}$ in $\mathscr{J}_S$ in such a way that the composition $\mathbb{T}_{g,m} \hookrightarrow \mathscr{J}_S \twoheadrightarrow \mathscr{J}_S/\mathscr{D}\!\mathit{iff}_0(S, \boldsymbol{z}) \cong \mathbb{T}_{g,m}$ is the identity map. This imbedding $\mathbb{T}_{g,m} \hookrightarrow \mathscr{J}_S$ is the desired slice. The choice of such imbedding $\mathbb{T}_{g,m} \hookrightarrow \mathscr{J}_S$ is equivalent to the choice of the complex structure $J_{\mathbb{T},S,\boldsymbol{z}}$ on the product $\mathbb{T}_{g,m} \times S$. One considers $\mathbb{T}_{g,m} \times S$ with this complex structure and with the holomorphic projection $\mathsf{pr} : \mathbb{T}_{g,m} \times S \to \mathbb{T}_{g,m}$ as the universal family corresponding to $\mathbb{T}_{g,m}$.

After the choice of the slice $\mathbb{T}_{g,m} \hookrightarrow \mathscr{J}_S$, $\mathscr{M}(\boldsymbol{x})$ is obtained as the quotient of the space

$$\widehat{\mathscr{M}}(\boldsymbol{x}) := \{(u, J_S, J) \in \mathscr{P}^*(S, \boldsymbol{z}; X, \boldsymbol{x}) : J_S \in \mathbb{T}_{g,m}\}$$

by the group $\mathbf{G}$ of biholomorphisms of $\mathbb{T}_{g,m} \times S$ preserving the projection $\mathsf{pr} : \mathbb{T}_{g,m} \times S \to \mathbb{T}_{g,m}$. It is discrete except the cases where $g = 0$ and $m = 1$ or 2. In the case $m = 1$, $S\backslash\{z_1\}$ is the complex plane $\mathbb{C}$ and $\mathbf{G}$ is its automorphism group $\mathbb{C}^* \ltimes \mathbb{C}$. Similarly, in the case $m = 2$, $S\backslash\{z_1, z_2\}$ is the punctured complex plane $\mathbb{C}^*$ and $\mathbf{G} = \mathbb{Z}_2 \ltimes \mathbb{C}^*$ is likewise its automorphism group. In either case one can construct a local slice for the action of $\mathbf{G}$ on $\widehat{\mathscr{M}}(\boldsymbol{x})$ by repeating the arguments of *Paragraph 2.2*. So the quotient $\widehat{\mathscr{M}}(\boldsymbol{x})/\mathbf{G}$ is a C^ℓ-smooth Banach manifold.

Now we define the notion of the normal sheaf of a pseudoholomorphic curve passing through fixed points on X. In this new situation, the linearization of $\overline{\partial}$-equations leads to the operator

$$D = D_{u,J} : \{v \in L^{1,p}(S, E_u) : v(z_i) = 0 \text{ for } i = 1, \ldots, m\} \to L^p_{(0,1)}(S, E), \tag{2.4.2}$$

which is the usual Gromov operator $D = D_{u,J}$, but now considered with a new domain of definition

$$\mathscr{E}_{u,\boldsymbol{x}} := \{v \in L^{1,p}(S, E_u) : v(z_i) = 0 \text{ for } i = 1, \ldots, m\}.$$

The space $\mathscr{E}_{u,\boldsymbol{x}}$ is the kernel of the evaluation homomorphism $\mathsf{ev}_z : \mathscr{E}_u \to E_{\boldsymbol{z}}$ and is the tangent plane to $\mathscr{S}(\boldsymbol{z}, \boldsymbol{x})$, see (2.4.1).

We now describe the structure of the operator (2.4.2). Recall that we have the decomposition $D_{u,J} = \overline{\partial}_{u,J} + R_{u,J}$, see *Paragraph 1.4*. Observe that the sheaf $\mathscr{O}(E_u)[-\boldsymbol{z}]$ of holomorphic sections of $\mathscr{O}(E_u)$ vanishing at the points $z_1, \ldots, z_m \in S$ is locally free and hence corresponds to a holomorphic bundle. Let us denote this bundle by $E_{u,-\boldsymbol{z}}$.

Lemma 2.4.1. *i) The (co)kernel of the operator*

$$\overline{\partial}_{u,J} : \{v \in L^{1,p}(S, E_u) : v(z_i) = 0 \textit{ for } i = 1, \ldots, m\} \to L^p_{(0,1)}(S, E), \tag{2.4.3}$$

is canonically isomorphic to the cohomology groups $\mathsf{H}^0_{\overline{\partial}}(S, E_{u,-\boldsymbol{z}})$ *and* $\mathsf{H}^1_{\overline{\partial}}(S, E_{u,-\boldsymbol{z}})$.

ii) The operator $D_{u,J}$ *induces the operator*

$$D_{u,-\boldsymbol{z},J} : L^{1,p}(S, E_{u,-\boldsymbol{z}}) \to L^p_{(0,1)}(S, E_{u,-\boldsymbol{z}})$$

which is of the form $D_{u,-\boldsymbol{z},J} = \overline{\partial}_{u,-\boldsymbol{z},J} + R_{u,-\boldsymbol{z},J}$, *where* $\overline{\partial}_{u,-\boldsymbol{z},J}$ *is the Cauchy-Riemann operator corresponding to the natural holomorphic structure in* $E_{u,-\boldsymbol{z}}$ *and* $R_{u,-\boldsymbol{z},J}$ *is a* $\mathbb{C}$*-antilinear* L^∞*-bounded bundle homomorphism, i.e.*

$$R_{u,-\boldsymbol{z},J} \in L^\infty\big(S, \overline{\mathsf{Hom}}_{\mathbb{C}}(E_{u,-\boldsymbol{z}}, E_{u,-\boldsymbol{z}} \otimes \Lambda^{(0,1)} S)\big).$$

iii) The (co)kernel of the operator

$$D_{u,J} : \{v \in L^{1,p}(S, E_u) : v(z_i) = 0 \textit{ for } i = 1, \ldots, m\} \to L^p_{(0,1)}(S, E) \tag{2.4.4}$$

is canonically isomorphic to the cohomology groups $\mathsf{H}^0_D(S, E_{u,-\boldsymbol{z}})$ *and* $\mathsf{H}^1_D(S, E_{u,-\boldsymbol{z}})$ *corresponding to the operator* $D_{u,-\boldsymbol{z},J}$.

Proof. Fix local holomorphic coordinates ζ_i on S, each centered at the corresponding marked point z_i. Consider the natural inclusions

$$j^0 : L^{1,p}(S, E_{u,-\boldsymbol{z}}) \hookrightarrow \{v \in L^{1,p}(S, E_u) : v(z_i) = 0 \text{ for } i = 1, \dots, m\}, \tag{2.4.5}$$

$$j^1 : L^p_{(0,1)}(S, E_{u,-\boldsymbol{z}}) \hookrightarrow L^p_{(0,1)}(S, E_u). \tag{2.4.6}$$

Observe that $v \in L^{1,p}(S, E_u)$ with $v(z_i) = 0$ belongs to $L^p_{(0,1)}(S, E_{u,-\boldsymbol{z}})$ if and only if locally near every z_i it has the form $v(\zeta_i) = \zeta_i w(\zeta_i)$ for some (uniquely defined!) $L^{1,p}$-section $w(\zeta_i)$ of $E_{u,-\boldsymbol{z}}$. This is equivalent to the condition $\zeta_i^{-1}\bar\partial_{u,J} v(\zeta_i) \in L^p$ as well as to the condition $\zeta_i^{-1} D_{u,J} v(\zeta_i) \in L^p$. Consequently, $D_{u,J}$ restricted to $L^{1,p}(S, E_{u,-\boldsymbol{z}})$ takes values in $L^p_{(0,1)}(S, E_{u,-\boldsymbol{z}})$. This yields the operator $D_{u,-\boldsymbol{z},J}$. Moreover, $D_{u,-\boldsymbol{z},J}$ is of order 1 and has the Cauchy-Riemann symbol. Consequently, it has the form $D_{u,-\boldsymbol{z},J} = \bar\partial_{u,-\boldsymbol{z},J} + R_{u,-\boldsymbol{z},J}$, where $\bar\partial_{u,-\boldsymbol{z},J}$ is the Cauchy-Riemann operator corresponding to the holomorphic structure in $E_{u,-\boldsymbol{z}}$, and $R_{u,-\boldsymbol{z},J}$ is the $\mathbb{C}$-antilinear part of $D_{u,-\boldsymbol{z},J}$.

Let $v_1(\zeta_i), \dots, v_n(\zeta_i)$ be a local holomorphic frame of E_u in a neighborhood of z_i and $R_{\alpha\beta}(\zeta_i)$ the matrix of $R_{u,J}$ in this frame. Then $\zeta_i v_1(\zeta_i), \dots, \zeta_i v_n(\zeta_i)$ is a local frame of $E_{u,-\boldsymbol{z}}$. From $\mathbb{C}$-antilinearity of $R_{u,J}$ we obtain

$$R_{u,J}(\zeta_i v_\alpha(\zeta_i)) = \textstyle\sum_\beta R_{\alpha\beta}(\zeta_i)\bar\zeta_i v_\beta(\zeta_i) = \sum_\beta \frac{\bar\zeta_i}{\zeta_i} R_{\alpha\beta}(\zeta_i)\cdot \zeta_i v_\beta(\zeta_i).$$

This shows that $\frac{\bar\zeta_i}{\zeta_i} R_{\alpha\beta}(\zeta_i)$ is the matrix of $R_{u,-\boldsymbol{z},J}$ in the frame $\zeta_i v_1(\zeta_i), \dots, \zeta_i v_n(\zeta_i)$. Recall that $R_{u,J}$ is a continuous bundle homomorphism (see *Lemma 1.4.2*, *i)*). So we see that $R_{u,-\boldsymbol{z},J}$ is also continuous outside the marked points z_i and has singularities of the form $\frac{\bar\zeta_i}{\zeta_i} R_{u,J}$ at z_i. In particular, $R_{u,-\boldsymbol{z},J}$ is of type L^∞, but is not continuous in general.

The equality of the kernels of the operators in *i)* and *ii)* with the corresponding 0-cohomology groups follows directly from the definition of the operators $\bar\partial_{u,-\boldsymbol{z},J}$ and $D_{u,-\boldsymbol{z},J}$. The equality for 1-cohomology groups will be shown only for the operator $D_{u,-\boldsymbol{z},J}$, the other one is carried out in the same manner. So let $\varphi \in L^p_{(0,1)}(S, E_{u,-\boldsymbol{z}})$. If $\varphi = D_{u,-\boldsymbol{z},J}(v)$ for $v \in L^{1,p}(S, E_{u,-\boldsymbol{z}})$, then $v \in L^{1,p}(S, E_u)$ and $j^1\varphi = D_{u,J}(v)$, or more precisely $j^1\varphi = D_{u,J}(j^0(v))$. This shows that the inclusion j^1 in (2.4.6) induces a well-defined homomorphism from $\mathsf{H}^1_D(S, E_{u,-\boldsymbol{z}})$ to the cokernel of (2.4.4). Moreover, $\varphi \in L^p_{(0,1)}(S, E_{u,-\boldsymbol{z}})$ induces the zero class in the cokernel of (2.4.4) if and only if $j^1(\varphi) = D_{u,J}(v)$ for some $v \in L^{1,p}(S, E_u)$ with $v(z_i) = 0$. But then locally

$$D_{u,J}(\zeta_i^{-1} v(\zeta_i)) = \zeta_i^{-1} D_{u,-\boldsymbol{z},J}(v(\zeta_i)) = \zeta_i^{-1} j^1\varphi(\zeta_i) \in L^p$$

by the definition of the inclusion $E_{u,-\boldsymbol{z}} \hookrightarrow E_u$. This implies that $v \in L^{1,p}(S, E_{u-\boldsymbol{z}})$. Thus the homomorphism induces by j^1 is injective.

Further, for any $\varphi \in L^p_{(0,1)}(S, E_u)$ there exists $v \in L^{1,p}(S, E_u)$ which vanishes at all z_i and solves the equation $\varphi = D_{u,J}(v)$ in a neighborhood of every z_i. Then $\varphi - D_{u,J}(v)$ represents the same class in the cokernel of (2.4.4) and is of the form $\varphi - D_{u,J}(v) = j^1(\psi)$ for some $\psi \in L^p_{(0,1)}(S, E_{u,-\boldsymbol{z}})$. This finishes the proof of the claim *iii)*. □

Now we define the normal sheaf of a pseudoholomorphic curve passing through fixed points. The construction is completely analogous to that in the case of no fixed points. Here, instead of the tangent bundle TS we use the bundle related the new situation. This is the bundle $TS_{-\boldsymbol{z}}$ associated to the locally free coherent sheaf $\mathscr{O}(TS)[-\boldsymbol{z}]$ of local holomorphic sections of TS vanishing at the points z_i. One can prove the analog of *Lemma 2.4.1* for $TS_{-\boldsymbol{z}}$. Observe however, that such a result follows immediately from that lemma if we set $X = S$ and $u = \mathsf{Id}_S$.

As in *Paragraph 1.5*, we obtain the sheaf homomorphism $du : \mathscr{O}(TS_{-\boldsymbol{z}}) \to \mathscr{O}(E_{u,-\boldsymbol{z}})$, which is injective for non-constant $u : S \to X$. Now, the *normal sheaf to curve $C = u(S)$ passing through the points $\boldsymbol{x} = (x_1, \dots, x_m)$* is defined as the quotient $\mathscr{N}_{u,\boldsymbol{x}} := \mathscr{O}(E_{u,-\boldsymbol{z}})/du\big(\mathscr{O}(TS_{-\boldsymbol{z}})\big)$ together with the exact sequence

$$0 \longrightarrow \mathscr{O}(TS_{-\boldsymbol{z}}) \xrightarrow{du} \mathscr{O}(E_{u,-\boldsymbol{z}}) \longrightarrow \mathscr{N}_{u,\boldsymbol{x}} \longrightarrow 0. \tag{2.4.7}$$

The sheaf $\mathscr{N}_{u,\boldsymbol{x}}$ can be decomposed into its *regular part* $\mathscr{N}^{\mathsf{reg}}_{u,\boldsymbol{x}}$ and its *singular part* $\mathscr{N}^{\mathsf{sing}}_{u,\boldsymbol{x}}$, where $\mathscr{N}^{\mathsf{reg}}_{u,\boldsymbol{x}}$ is locally free and $\mathscr{N}^{\mathsf{sing}}_{u,\boldsymbol{x}}$ is a torsion sheaf. Then $\mathscr{N}^{\mathsf{reg}}_{u,\boldsymbol{x}}$ is a sheaf of local holomorphic sections of the *normal bundle $N_{u,\boldsymbol{x}}$ to curve $C = u(S)$ passing through the points $\boldsymbol{x} = (x_1, \dots, x_m)$*, so that $\mathscr{N}^{\mathsf{reg}}_{u,\boldsymbol{x}} = \mathscr{O}(N_{u,\boldsymbol{x}})$.

As in *Paragraph 1.5*, we also obtain the exact sequence

$$0 \longrightarrow \mathscr{O}(TS_{-\boldsymbol{z}}) \otimes \mathscr{O}([A]) \xrightarrow{du} \mathscr{O}(E_{u,-\boldsymbol{z}})\mathscr{O}(N_{u,\boldsymbol{x}}) \longrightarrow 0. \tag{2.4.8}$$

where $[A]$ is the branching divisor of du (see *Definition 1.5.1*). This implies that the regular part $\mathscr{O}(N_{u,\boldsymbol{x}})$ is the quotient

$$\mathscr{O}(N_{u,\boldsymbol{x}}) = \mathscr{O}(E_{u,-\boldsymbol{z}})/du\big(\mathscr{O}(TS_{-\boldsymbol{z}}) \otimes \mathscr{O}([A])\big).$$

From the definition of $E_{u,-\boldsymbol{z}}$ and $TS_{-\boldsymbol{z}}$ we obtain the isomorphism

$$\mathscr{O}(N_{u,\boldsymbol{x}}) \cong \mathscr{O}(N_u) \otimes \mathscr{O}([A]).$$

On the other hand, the singular part remains the same as is the case without constraints:

$$\mathscr{N}^{\mathsf{sing}}_{u,\boldsymbol{x}} \cong \mathscr{N}^{\mathsf{sing}}_{u} \cong \mathscr{O}/\mathscr{O}(-[A]).$$

Further, we observe that the operators $\overline{\partial}$ on $TS_{-\boldsymbol{z}}$ and $D_{u,-\boldsymbol{z},J}$ in $E_{u,-\boldsymbol{z}}$ commute with the homomorphism $du : TS_{-\boldsymbol{z}} \to E_{u,-\boldsymbol{z}}$. Consequently, $D_{u,-\boldsymbol{z},J}$ induces the operator

$$D^N_{u,-\boldsymbol{z},J} : L^{1,p}(S, N_{u,-\boldsymbol{x}}) \to L^p_{0,1)}(S, N_{u,-\boldsymbol{x}})$$

with the properties similar to ones of (1.5.3). Further, as in *Lemma 1.5.3* and *Corollary 1.5.4* we obtain a long exact sequence of D-cohomologies.

Proposition 2.4.2. *The short exact sequence* (2.4.7) *induces the long exact sequence of D-cohomologies*

$$\begin{array}{cccccccc} 0 & \longrightarrow & \mathsf{H}^0(S, TS_{-\boldsymbol{z}}) & \longrightarrow & \mathsf{H}^0_D(S, E_{u,-\boldsymbol{z}}) & \longrightarrow & \mathsf{H}^0_D(S, N_{u,-\boldsymbol{z}}) \oplus \mathsf{H}^0(S, \mathscr{N}^{\mathsf{sing}}_u) & \xrightarrow{\delta} \\ & \longrightarrow & \mathsf{H}^1(S, TS_{-\boldsymbol{z}}) & \longrightarrow & \mathsf{H}^1_D(S, E_{u,-\boldsymbol{z}}) & \longrightarrow & \mathsf{H}^1_D(S, N_{u,-\boldsymbol{z}}) & \longrightarrow \; 0. \end{array}$$

Finally, we note that the results of *Paragraphs 2.2* and *2.3* remain valid, after an appropriate modification, also for curves passing through fixed points. We state without the proof the summary of results which will be used later.

Theorem 2.4.3. *i) The total moduli space $\mathscr{M}_{\boldsymbol{x}}$ of pseudoholomorphic curves in a given homology class $[C] \in \mathsf{H}_2(X, \mathbb{Z})$ passing through fixed pairwise distinct points $\boldsymbol{x} = (x_1, \dots, x_m)$ on X is a C^ℓ-smooth Banach submanifold of $\mathscr{M}$ of real codimension $2m$. In particular, the projection $\pi_{\mathscr{J}} : \mathscr{M}_{\boldsymbol{x}} \to \mathscr{J}$ is a C^ℓ-smooth Fredholm map of index*

$$2(c_1(X)[C] + (n-3)(1-g) - m).$$

ii) For a generic $J \in \mathscr{J}$ and a generic C^ℓ-smooth path $h : [0,1] \to \mathscr{J}$ the fiber

$$\mathscr{M}_{J,\boldsymbol{x}} := \pi^{-1}_{\mathscr{J}}(J)$$

and the relative moduli space

$$\mathscr{M}_{h,\boldsymbol{x}} := [0,1] \times_{\mathscr{J}} \mathscr{M}_{\boldsymbol{x}}$$

are C^ℓ-smooth manifolds of expected dimension $2(c_1(X)[C] + (n-3)(1-g) - m)$ *and* $2(c_1(X)[C] + (n-3)(1-g) - m) + 1$ *respectively.*

3. Cusp-curves in the moduli space.

In this section we study the problem of deformation of pseudoholomorphic curves with prescribed singularities and develop the techniques required for controlling their singularities under deformation. As the main result of this section we show that the locus of pseudoholomorphic curves with a prescribed type of singularity is a smooth Banach submanifold of expected codimension in the total moduli space of pseudoholomorphic curves. This improvement of the result of Micallef and White (see *Lemma 1.2.1*) plays a crucial role below in *Section 4* in the proof of the *saddle point property*.

Recall that our moduli space $\mathscr{M}$ consists of parameterized non-multiple pseudoholomorphic curves, *i.e.* pseudoholomorphic maps from a fixed real surface S modulo reparameterizations.

Definition 3.0.1. A point $z \in S$ on a J-holomorphic curve $u : S \to X$ is a *cusp*, or a *cuspidal point*, if $\mathsf{ord}_z du > 0$. The number $\mathsf{ord}_z du$ is called the *order of the cusp* of u at z. A J-holomorphic curve $u : S \to X$ containing cuspidal points is called a *cusp curve*.

Note that in the literature on pseudoholomorphic curves the notion "cusp curve" has a different meaning. Our terminology agrees rather with the one used in algebraic geometry where the notion "cusp" means a "peak", *i.e.* an irreducible singularity. This describes the situation at hand more accurately.

3.1. **Deformation of pseudoholomorphic maps.** Explicit construction of deformations is needed to obtain local charts for subspaces of curves with prescribed singularities.

Lemma 3.1.1. *Let $B \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$ be the unit ball, $\mathscr{Y}$ a Banach manifold, $\{J_{\eta,t}\}_{\eta\in\mathscr{Y},t\in[0,1]}$ a family of homotopies of almost complex structures in B with parameterized by $\mathscr{Y}$ and depending $C^{\ell-1}$-smoothly on $(\eta,t) \in \mathscr{Y} \times [0,1]$. Further, let $u_{\eta,0} : \Delta \to B$, $\eta \in \mathscr{Y}$, be a $C^{\ell-1}$-smooth family of $J_{\eta,0}$-holomorphic map, such that $u_{\eta,0}(\Delta) \subset B(\frac{1}{2})$, and $\mathsf{ord}_0(du_{\eta,t}) = \mu$.*

Then for any family $v_\eta \in \mathbb{R}^{2n}$ depending $C^{\ell-1}$-smoothly on $\eta \in \mathscr{Y}$ and any $\nu \in \mathbb{N}$ there exists $t^* = t^*(J_t, u_0, v, \mu, \nu) > 0$, a neighborhood $U_{\mathscr{Y}}$ of a given $\eta^* \in \mathscr{Y}$, and a $C^{\ell-1}$-smooth family of homotopies $\{w_{\eta,t}\}_{\eta\in U_{\mathscr{Y}},t\in[0,t^*]}$ with $w_{\eta,t} \in L^{1,p}(\Delta, \mathbb{R}^{2n})$ such that the maps $u_{\eta,t} : \Delta \to B$ given by

$$u_{\eta,t}(z) = u_{\eta,0}(z) + z^\nu (t\, v_\eta + w_{\eta,t}(z)) \tag{3.1.1}$$

i) are $J_{\eta,t}$-holomorphic if $\nu \leqslant 2\mu + 1$, and
ii) are $J_{\eta,0}$-holomorphic if $\nu > 2\mu + 1$.

Moreover, for $z \neq 0$ the function $w_{\eta,t}(z)$ depends $C^{\ell-1}$-smoothly on (η, t, z).

Remarks. 1. In other words, there exists a pseudoholomorphic deformation u_t of a given map u_0 in a given direction $\frac{d}{dt} u_t|_{t=0} = z^\nu v + O(|z|^{\nu+\alpha})$; and moreover, for smaller ν it is possible to deform simultaneously the almost complex structure. Furthermore, if the initial data depend smoothly on the parameter η, then the corresponding constructions give a smooth dependence of the maps on η.

2. The loss of smoothness from C^ℓ to $C^{\ell-1}$ is due to the fact that the Gromov operator $D_{u,J}$ depends only $C^{\ell-1}$-smoothly on u. Indeed, $D_{u,J}$ is the derivative of the $\overline{\partial}$-operator $u \mapsto \overline{\partial}_J u$ in the u-direction, which is only C^ℓ-smooth.

Proof. We give only a sketch. First, we fix a family $\varphi_{\eta,t}$ of affine transformations of $\mathbb{R}^{2n}$ with depend $C^{\ell-1}$-smoothly on (η,t) such that $\varphi_{\eta,t}\circ u_{\eta,0}(0)=0\in B$ and $\varphi_{\eta,t}\circ J_{\eta,t}\circ\varphi_{\eta,t}^{-1}$ coincide with J_{st} in $u_{\eta,0}(0)$. Setting $\tilde u_{\eta,0} := \varphi_{\eta,t}\circ u_{\eta,0}$ and $\tilde J_{\eta,t} := \varphi_{\eta,t}\circ J_{\eta,t}\circ\varphi_{\eta,t}^{-1}$ we reduce the problem to the case where $\tilde u_{\eta,0}(0)=0$ and $\tilde J_{\eta,t}(0)=J_{\mathsf{st}}$.

Now we assume that there is no dependence on the parameter and drop the index η. Using (3.1.1) one writes the equation $\overline{\partial}_{J_t} u_t = 0$ in the form

$$(x+yJ_{\mathsf{st}})^{-\nu}\overline{\partial}_{J_t}\big(u_0(z)+(x+yJ_{\mathsf{st}})^{\nu}(tv+w_t(z))\big)=0 \tag{3.1.2}$$

with $x+\mathrm{i}y=z$ the standard coordinates on Δ, and considers (3.1.2) as an equation for $w_t(z)$. Then one shows that under the hypotheses of the lemma the linearization of (3.1.2) has the form

$$(\overline{\partial}^{(\nu)}_{u_t,J_t}+R^{(\nu)}_{u_t,J_t})\dot w_t(z)=\psi^{(\nu)}_{u_t,J_t}(\dot J_t)(z), \tag{3.1.3}$$

where $\dot w_t(z)=\frac{d}{dt}w_t(z)$ and $\psi_t^{(\nu)}(\dot J_t)(z)\in L^\infty(\Delta,\mathbb{C}^n)$. Thus it is sufficient to find a right inverse $T^{(\nu)}_{u_t,J_t}$ of the Gromov type operator $D^{(\nu)}_{u_t,J_t}=\overline{\partial}^{(\nu)}_{u_t,J_t}+R^{(\nu)}_{u_t,J_t}$ with an additional condition $\dot w_t(0)=0$. We refer to [Iv-Sh-1], *Lemma 3.3.1*, for the explicit construction of such a right inverse $T^{(\nu)}_{u,J}$. Moreover, the operator $T^{(\nu)}_{u,J}$ and the inhomogeneity term $\psi^{(\nu)}_{u,J}$ depend smoothly on u and J. As a consequence, the solution w_t of (3.1.2) depends $C^{\ell-1}$-smoothly on the parameter $\eta\in\mathscr{Y}$. □

Definition 3.1.1. Let $B\subset\mathbb{R}^{2n}\cong\mathbb{C}^n$ be a ball, J_0 a C^ℓ-smooth almost complex structure in B, $u_0:\Delta\to B$ a J_0-holomorphic map and $\nu\geqslant 1$ an integer exponent. Denote by $\mathsf{dfrm}_\nu(u,J;v)$ the a map depending $C^{\ell-1}$-smoothly on

- a C^ℓ-smooth almost complex structure J in B, sufficiently close to J_0;
- a J-holomorphic map u, sufficiently close to u_0;
- a vector $v\in\mathbb{R}^{2n}$, sufficiently close to 0;

such that $\tilde u := \mathsf{dfrm}_\nu(u,J;v)$ is a J-holomorphic map of the form $\tilde u(z)=u(z)+z^\nu v+O(|z|^{\nu+\alpha})$. Note that the choice of such a map dfrm_ν is not unique.

Lemma 3.1.1 allows us to construct local deformations of pseudoholomorphic maps with appropriate types of singularities. To obtain a global deformation, we use

Lemma 3.1.2. *Let $u_0:S\to X$ be a non-multiple J_0-holomorphic map, $z_1,\dots,z_m$ fixed points on S, and $U_1,\dots,U_m\subset S$ disjoint neighborhoods of these points. Further, let $\{J_t\}_{t\in[0,1]}$ be a given $C^{\ell-1}$-smooth homotopy of almost complex structures on X, and $\{u_{i,t}\}_{t\in[0,1]}$ given $C^{\ell-1}$-smooth homotopies of J_t-holomorphic maps $u_{i,t}:U_i\to X$.*

Then there exist $t^*>0$, a $C^{\ell-1}$-smooth homotopy $\{\tilde J_t\}_{t\in[0,t^*]}$ of almost complex structures on X, a $C^{\ell-1}$-smooth homotopy $\{\tilde u_t\}_{t\in[0,t^*]}$ of $\tilde J_t$-holomorphic maps $\tilde u_t:S\to X$ such that u_t coincides with each $u_{i,t}$ in some (possibly smaller) neighborhood of z_i and $\tilde J_t$ coincides with J_t in some neighborhood of each $x_i:=u_0(z_i)$.

The proof of the lemma is left to the reader.

Refining the result of *Lemma 3.1.1* we show that the condition $u_1(z)-u_2(z)=o(|z|^k)$ of *Lemma 1.2.5* defines a submanifold in the spaces of pairs of pseudoholomorphic maps.

Definition 3.1.2. Define *the spaces of pairs of pseudoholomorphic maps coinciding up to order k at $z=0$* as $\mathscr{PP}_k(\Delta,X) :=$

$$\big\{(u',u'',J)\in L^{1,p}(\Delta,X)\times L^{1,p}(\Delta,X)\times \mathscr{J} : \bar\partial_J u' = 0 = \bar\partial_J u'', u'(z)-u''(z) = o(|z|^k)\big\}, \tag{3.1.4}$$

where the condition $u'(z)-u''(z)=o(z^k)$ is related to any local coordinate system on X in a neighborhood of the point $u'(0)=u''(0)\in X$.

The structure of $\mathscr{PP}_m(\Delta,X)$ for the cases $k=0$ and $k=1$ is easily obtained from transversality techniques. In general we have

Theorem 3.1.3. *Assume that $\mathscr{J}$ consists of C^ℓ-smooth structures with $\ell\geqslant 2$. Then the space $\mathscr{PP}_k(\Delta,X)$ is a $C^{\ell-1}$-submanifold of the fiber product $\mathscr{P}(\Delta,X)\times_{\mathscr{J}}\mathscr{P}(\Delta,X)$ of codimension of $2n(k+1)$ with the the tangent space*

$$T_{(u',u'',J)}\mathscr{PP}_k(\Delta,X) = \big\{(v',v'',\dot J)\in T_{(u',u'',J)}\big(\mathscr{P}(\Delta,X)\times_{\mathscr{J}}\mathscr{P}(\Delta,X)\big) : j^k(v'-v'')=0\big\}. \tag{3.1.5}$$

Moreover, for $k=0$ and $k=1$ the space $\mathscr{PP}_k(\Delta,X)$ is well-defined and C^ℓ-smooth also for $\ell\geqslant 1$.

Proof. It follows from *Lemma 2.2.4* that $\mathscr{P}(\Delta,X)\times_{\mathscr{J}}\mathscr{P}(\Delta,X)$ is a C^ℓ-smooth Banach manifold with the tangent space

$$T_{(u',u'',J)}\big(\mathscr{P}(\Delta,X)\times_{\mathscr{J}}\mathscr{P}(\Delta,X)\big) = \big\{(v',v'',\dot J) : (v',\dot J)\in T_{(u',J)}\mathscr{P}(\Delta,X), (v'',\dot J)\in T_{(u'',J)}\mathscr{P}(\Delta,X)\big\}, \tag{3.1.6}$$

so that $D_{u',J}v' + \dot J\circ du'\circ J_\Delta = 0$ and similarly for v''.

Fix $(u'_0,u''_0,J_0)\in\mathscr{PP}_0(\Delta,X)$ and local coordinates (w_i) in a neighborhood $U\subset X$ of $x^* := u'_0(0)=u''_0(0)\in X$. Then there exists $r>0$ such that for any pair (u',u'') of $L^{1,p}(\Delta,X)$-maps sufficiently close to (u'_0,u''_0) we have $u'(\Delta(r))\subset U$ and $u''(\Delta(r))\subset U$. The coordinates in U induce the linear structure. Thus we can consider the difference $u'(z)-u''(z)$ having in mind that it is well-defined only for $z\in\Delta(r)$.

The subspace $\mathscr{PP}_0(\Delta,X)$ is defined by the condition $u'(0)=u''(0)$ for $(u',u'',J)\in\mathscr{P}(\Delta,X)\times_{\mathscr{J}}\mathscr{P}(\Delta,X)$. Setting $F(u',u'',J) := u'(0)-u''(0)$ we obtain a C^ℓ-smooth function, which is well-defined in a neighborhood of (u'_0,u''_0,J_0) and is a local defining function for $\mathscr{PP}_0(\Delta,X)$. The differential of F in $(u',u'',J)\in\mathscr{PP}_0(\Delta,X)$,

$$dF : T_{(u',u'',J)}\big(\mathscr{P}(\Delta,X)\times_{\mathscr{J}}\mathscr{P}(\Delta,X)\big)\to T_{u'(0)}X,$$

is given by the formula $dF(v',v'',\dot J) = v'(0)-v''(0)$ and is a surjective map. Thus $\mathscr{PP}_0(\Delta,X)$ is a C^ℓ-smooth submanifold of $\mathscr{P}(\Delta,X)\times_{\mathscr{J}}\mathscr{P}(\Delta,X)$ of codimension $2n=\mathsf{dim}_{\mathbb{R}}X$.

Considering the C^ℓ-smooth map $\mathsf{ev}_0 : \mathscr{PP}_0(\Delta,X)\to X$ with

$$\mathsf{ev}_0(u',u'',J) := u'(0)=u''(0),$$

we obtain a C^ℓ-smooth bundle $E^{(0)}$ over $\mathscr{PP}_0(\Delta,X)$ with fiber $E^{(0)}_{(u',u'',J)} = u'^*T_{u'(0)}X$. The formulas $\sigma'(u',u'',J) := du'(0)$ and $\sigma''(u',u'',J) := du''(0)$ define C^ℓ-smooth sections of the bundle $T^*_0\Delta\otimes E^{(0)}$ over $\mathscr{PP}_0(\Delta,X)$. Thus the condition $du'(0)=du''(0)$ is equivalent to the vanishing of $\sigma'-\sigma''$. Consequently, $\mathscr{PP}_1(\Delta,X)$ is a C^ℓ-smooth submanifold of $\mathscr{PP}_0(\Delta,X)$ of codimension $2n$.

We proceed further by induction using the case $k=0$ as the base. Our notation is as follows. For a triple $(u',u'',J) \in \mathscr{P}\mathscr{P}_0(\Delta, X)$ we consider the (integrable) complex structure J_{st} in U with coincides with J at the point $u'(z)=u''(z)$ and is constant with respect to the coordinates in U. Note that J_{st} depends C^ℓ-smoothly on $(u',u'',J) \in \mathscr{P}\mathscr{P}_k(\Delta, X)$. Thus we can regard U as an open subset in $\mathbb{C}^n$.

For a pair (u',u'') of J-holomorphic maps with values in $U \subset X$ we obtain

$$
\begin{aligned}
0 = \overline{\partial}_J u' - \overline{\partial}_J u'' &= \big((\partial_x u' - J(u')\cdot\partial_y u') - (\partial_x u'' - J(u'')\cdot \partial_y u'')\\
&= \partial_x(u'-u'') + J(u')\cdot\partial_y(u'-u'') + \big(J(u') - J(u'')\big)\cdot\partial_y u''\\
&= \overline{\partial}_{J(u')}(u'-u'') + \big(J(u')-J(u'')\big)\cdot\partial_y u''. \qquad (3.1.7)
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\overline{\partial}_{J_{\mathsf{st}}}(u'-u'') &= \overline{\partial}_{J_{\mathsf{st}}}(u'-u'') - (\overline{\partial}_J u' - \overline{\partial}_J u'')\\
&= \big(J_{\mathsf{st}} - J(u')\big)\cdot\partial_y(u'-u'') - \big(J(u')-J(u'')\big)\cdot\partial_y u''. \qquad (3.1.8)
\end{aligned}
$$

Let us denote the last expression by $H_{u',u'',J}(z)$

Now suppose that (u',u'',J) varies in $\mathscr{P}\mathscr{P}_k(\Delta,X)$ with $k \geqslant 1$. We can assume by induction that $\mathscr{P}\mathscr{P}_k(\Delta,X)$ is a $C^{\ell-1}$-smooth manifold. We claim that for any $p<\infty$

$$
f_k(z) := z^{-(k+1)}(u'(z)-u''(z)) \qquad (3.1.9)
$$

is a well-defined $L^{1,p}(\Delta(r),\mathbb{C}^n)$-valued function depending $C^{\ell-1}$-smoothly on $(u',u'',J) \in \mathscr{P}\mathscr{P}_k(\Delta,X)$. The claim implies the theorem. Indeed, the function F_k given by $F_k : (u',u'',J) \in \mathscr{P}\mathscr{P}_k(\Delta,X) \mapsto f_k(0) \in \mathbb{C}^n$ is then a local defining function for $\mathscr{P}\mathscr{P}_{k-1}(\Delta,X)$ inside $\mathscr{P}\mathscr{P}_k(\Delta,X)$, whereas non-degeneracy of dF_k can be easily obtained from *Lemma 3.1.1*.

Again by induction, we can suppose that $f_{k-1}(z) = z^{-k}(u'(z)-u''(z))$ is a well-defined $L^{1,p}(\Delta(r),\mathbb{C}^n)$-valued function depending $C^{\ell-1}$-smoothly on $(u',u'',J) \in \mathscr{P}\mathscr{P}_{k-1}(\Delta,X)$. Note that $f_{k-1}(0)$ vanishes identically on $\mathscr{P}\mathscr{P}_k(\Delta,X)$. Further, for any exponents p,p' with $2<p'<p<\infty$ the map $f(z) \in L^{1,p}(\Delta,\mathbb{C}^n) \mapsto z^{-1}(f(z)-f(0)) \in L^{p'}(\Delta,\mathbb{C}^n)$ is linear and bounded. Consequently, for any $p<\infty$ the function $f_k(z) = z^{-1}f_{k-1}(z)$ lies in $L^p(\Delta,\mathbb{C}^n)$ and depends $C^{\ell-1}$-smoothly on $(u',u'',J) \in \mathscr{P}\mathscr{P}_k(\Delta,X)$ with respect to the L^p-topology.

Without loss of generality we may assume that U is convex. The identity

$$
J(w) = J(w^*) + \int_{t=0}^1 \partial_t J(w^* + t(w-w^*))dt
$$

for $(w,w^*) \in U\times U$ implies the relation $J(w) = J(w^*) + \sum_i (w_i - w_i^*)S_i(w,w^*) = S(w,w^*; w-w^*)$ with the function $S(w,w^*;\tilde w)$ depending C^ℓ-smoothly on $J \in \mathscr{J}$, $C^{\ell-1}$-smoothly on $(w,w^*) \in U\times U$ and $\mathbb{R}$-linearly on $\tilde w \in \mathbb{C}^n$. Substituting $u''(z) = u'(z) + z^{k+1}f_k(z)$ in $(J(u')-J(u''))\cdot\partial_y u''$ we obtain

$$
(J(u'(z)) - J(u''(z)))\cdot\partial_y u''(z) = S\big(u''(z),u'(z); z^{k+1}f_k(z)\big)\cdot\partial_y u''(z)
$$

By apriori regularity estimates, for $r<1$ we can consider $du''(z)$ as a map from $\mathscr{P}\mathscr{P}_k(\Delta,X)$ to $C^0(\Delta(r),\mathbb{C}^n)$ which depends C^ℓ-smoothly on (u',u'',J). Thus we have represented the term $(J(u')-J(u''))\cdot\partial_y u''$ as a composition of the $C^{\ell-1}$-smooth map

$$
(u',u'',J) \in \mathscr{P}\mathscr{P}_k(\Delta,X) \;\mapsto\; S(u''(z),u'(z);f_k(z))\cdot\partial_y u''(z) \in L^p(\Delta(r),\mathbb{C}^n)
$$

and the linear bounded map

$$
S(u''(z),u'(z);f_k(z))\partial_y u''(z) \;\mapsto\; z^{-(k+1)}\cdot S(u''(z),u'(z);z^{k+1}\cdot f_k(z))\partial_y u''(z).
$$

Thus $(J(u')-J(u''))\cdot\partial_y u''$ depends C^ℓ-smoothly on $(u',u'',J)\in\mathscr{PP}_k(\Delta,X)$ with respect to the norm topology in $L^p(\Delta(r),\mathbb{C}^n)$. Consequently, the formula

$$(u',u'',J)\in\mathscr{PP}_k(\Delta,X)\ \mapsto\ z^{-(k+1)}\cdot(J_{\mathsf{st}}-J(u'(z)))\cdot\partial_y(u'(z)-u''(z))$$

defines a $L^p(\Delta(r),\mathbb{C}^n)$-valued map depending $C^{\ell-1}$-smoothly on $(u',u'',J)\in\mathscr{PP}_k(\Delta,X)$.

Similar estimates can be be carried out for the first term $(J_{\mathsf{st}}-J(u'))\cdot\partial_y(u'-u'')$ in (3.1.8). Together, this implies that $h_k(z):=z^{-k}H_{u',u'',J}(z)$ lies in $L^p(\Delta(r),\mathbb{C}^n)$ and depends $C^{\ell-1}$-smoothly on $(u',u'',J)\in\mathscr{PP}_k(\Delta,X)$ with respect to L^p-topology. Now let $f_{\overline{\partial},k}(z)$ be a solution of the equation $\overline{\partial}_{J_{\mathsf{st}}}f_{\overline{\partial},k}(z)=h_k(z)$ depending $C^{\ell-1}$-smoothly on $(u',u'',J)\in\mathscr{PP}_k(\Delta,X)$ with respect to the $L^{1,p}$-topology. Then $(u'(z)-u''(z))-z^{k+1}f_{\overline{\partial},k}(z)$ is a holomorphic $\mathbb{C}^n$-valued function, depending $C^{\ell-1}$-smoothly on $(u',u'',J)\in\mathscr{PP}_k(\Delta,X)$ with respect to $L^{1,p}$-topology and vanishing in $z=0$ up to order $k+1$. Consequently,

$$(u'(z)-u''(z))-z^{k+1}f_{\overline{\partial},k}(z)=z^{k+1}f_{\mathscr{O},k}(z)$$

and $f_k(z)=f_{\mathscr{O},k}(z)+f_{\overline{\partial},k}(z)$ possesses the property claimed above. □

3.2. **Curves with prescribed cusp order.** In this paragraph we show that J-curves with cusps of given order form a Banach submanifold of the moduli space and compute its codimension.

Definition 3.2.1. For a given natural m we denote by $\boldsymbol{k}$ an m-tuple $(k_1,\dots,k_m)$ with $k_i\geqslant 1$ and set $|\boldsymbol{k}|:=\sum_i k_i$. The m-tuple $(1,\dots,1)$ is denoted $\mathbf{1}_m$. Define the *moduli space $\mathscr{M}_{\boldsymbol{k}}$ of pseudoholomorphic curves with a given cusp order $\boldsymbol{k}$* as the set of classes $[u,J,\boldsymbol{z}]$ such that $[u,J]\in\mathscr{M}$ and u has m (marked) cusp-points $\boldsymbol{z}=\{z_1^*,\dots,z_m^*\}$ with $\mathsf{ord}_{z_i^*}\geqslant k_i$. Two triples $(u,J,\boldsymbol{z})$ and $(\tilde u,\tilde J,\tilde{\boldsymbol{z}})$ define the same class $[u,J,\boldsymbol{z}]=[\tilde u,\tilde J,\tilde{\boldsymbol{z}}]\in\mathscr{M}_{\boldsymbol{k}}$ if and only if there exists $g\in\mathbf{G}$ such that $\tilde u=u\circ g$ and $\tilde z_i^*=g(z_i^*)$.

The main result of this paragraph is

Theorem 3.2.1. *The set $\mathscr{M}_{\boldsymbol{k}}$ is a C^ℓ-smooth manifold and the natural map $\mathscr{M}_{\boldsymbol{k}}\longrightarrow\mathscr{M}$ given by $[u,J,\boldsymbol{z}]\mapsto[u,J]$ of $\mathscr{M}$ is an immersion of codimension $2(n|\boldsymbol{k}|-m)$, where $n=\mathsf{dim}_{\mathbb{C}}X$ and m is the number of marked cusp-points.*

We divide the proof in several steps. First we consider the corresponding problem for $\widehat{\mathscr{M}}$. The reason is that it is more convenient to work with maps, *i.e.* elements of $\widehat{\mathscr{M}}$, than with parameterized curves, *i.e.* elements of $\mathscr{M}$. This means that we are interested in the set

$$\widehat{\mathscr{M}_{\boldsymbol{k}}}:=\left\{(u,J_S,J;z_1^*,\dots z_m^*)\in\widehat{\mathscr{M}}\times(S)^m:\begin{array}{c}z_i^*\text{ are pairwise distinct,}\\ \mathsf{ord}_{z_i^*}du\geqslant k_i\end{array}\right\},\tag{3.2.1}$$

where $(S)^m=S\times\cdots\times S$ is the m-fold product of S. Obviously, the projection from $\widehat{\mathscr{M}}\times(S)^m$ onto $\widehat{\mathscr{M}}$ and then onto $\mathscr{M}$ maps $\widehat{\mathscr{M}_{\boldsymbol{k}}}$ onto $\mathscr{M}_{\boldsymbol{k}}$. In our proof of *Theorem 3.2.1* we shall show that this map $\widehat{\mathscr{M}_{\boldsymbol{k}}}\to\mathscr{M}_{\boldsymbol{k}}$ is a principle $\mathbf{G}$-bundle.

Definition 3.2.2. Set

$$\widehat{\mathscr{M}^{(m)}}:=\{(u,J_S,J;z_1^*,\dots z_m^*)\in\widehat{\mathscr{M}}\times(S)^m:z_i^*\neq z_j^*\text{ for every }i\neq j\}\tag{3.2.2}$$

denoting by S_i the i-th factor in $(S)^m$. Equip $\widehat{\mathscr{M}^{(m)}}$ with the maps $\mathsf{ev}_i:\widehat{\mathscr{M}^{(m)}}\to X^m$ defined by $\mathsf{ev}_i(u,J_S,J;z_1^*,\dots,z_m^*):=u(z_i^*)$. Denote by E_i the pulled-back bundles ev_i^*TX and $\mathsf{ev}^{(m)*}T(X^m)$ over $\widehat{\mathscr{M}^{(m)}}$. The fiber of E_i over $(u,J_S,J;\boldsymbol{z})$ is $(E_i)_{(u,J_S,J;\boldsymbol{z})}=T_{u(z_i^*)}X$.

Obviously, the space $\widehat{\mathscr{M}}^{(m)}$ is a C^ℓ-smooth Banach manifold, $\mathsf{ev}_i : \widehat{\mathscr{M}}^{(m)} \to X^m$ are C^ℓ-smooth maps, and E_i are C^ℓ-smooth bundles over $\widehat{\mathscr{M}}^{(m)}$. Note that we also have line bundles TS_i and T^*S_i over $\widehat{\mathscr{M}}^{(m)}$ which are defined in an obvious way as the (co)tangent bundles to each S_i.

Lemma 3.2.2. *The formula $\Upsilon(u, J_S, J; z_1^*, \dots z_m^*) := (du(z_1^*), \dots, du(z_m^*)) \in \bigoplus_i T^*S_i \otimes E_i$ defines a C^ℓ-smooth section of $\bigoplus_i T^*S_i \otimes E_i$ over $\widehat{\mathscr{M}}^{(m)}$, transversal to the zero section. The zero-set of Υ coincides with the space $\widehat{\mathscr{M}}_{\mathbf{1}_m}$ of maps having cups in each marked z_i^*. Thus $\widehat{\mathscr{M}}_{\mathbf{1}_m}$ is a C^ℓ-smooth Banach submanifold of $\widehat{\mathscr{M}}^{(m)}$ of codimension $2nm$.*

Before starting the proof we introduce some new notation.

Definition 3.2.3. Let $\mathscr{Y}$ be a C^ℓ-smooth Banach manifold and $f : \mathscr{Y} \to \mathbb{T}_g \times S$ a C^ℓ-smooth map of the form $f(y) = (J_S(y), z^*(y))$. Set $F(y) := (y, z^*(y))$ so that $F : \mathscr{Y} \to \mathscr{Y} \times S$ is an imbedding. A *local $J_S(y)$-holomorphic coordinate (or simply a J_S-holomorphic coordinate) on $\mathscr{Y} \times S$ centered at z^** is a C^ℓ-smooth $\mathbb{C}$-valued function z defined in some neighborhood $U \subset \mathscr{Y} \times S$ of $F(\mathscr{Y})$ which vanishes along $F(\mathscr{Y})$ and is $J_S(y)$-holomorphic along each $\{y\} \times S$. One can use *Lemma 3.1.1* for a proof of the existence of such a local holomorphic coordinate.

Proof of *Lemma 3.2.2*. It is obvious that Υ is well-defined. To show the C^ℓ-smoothness of Υ, for any $i = 1, \dots, m$, we fix some local coordinate z_i on $\widehat{\mathscr{M}}^{(m)}$ which is J_S-holomorphic along S_i and centered at $z_i^* \in S_i$. Now we can find a local frame $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$ of $T^*S_i \otimes E_i$ which depends C^ℓ-smoothly on $(u, J_S, J) \in \widehat{\mathscr{M}}$ and holomorphically on the coordinate z_i. The existence of such a frame follows from *Definition 1.4.1* and a parametric version of *Lemma 1.4.1*. The coefficients of $du \in T^*S_i \otimes E_i$ with respect to such a frame $\boldsymbol{\xi}$ depend C^ℓ-smoothly on $(u, J_S, J) \in \widehat{\mathscr{M}}$ and holomorphically on z_i. Consequently, the $du(z_i^*)$ depend C^ℓ-smoothly on $(u, J_S, J; \boldsymbol{z}) \in \widehat{\mathscr{M}}^{(m)}$. Thus Υ is C^ℓ-smooth.

The transversality of Υ to the zero-section of $\bigoplus_i T^*S_i \otimes E_i$ follows immediately from results of *Paragraph 3.1*. In particular, $\widehat{\mathscr{M}}^{(m)}_{\mathbf{1}_m}$ is the C^ℓ-smooth Banach submanifold of $\widehat{\mathscr{M}}^{(m)}$. The corresponding codimension is $\mathsf{rank}_{\mathbb{R}} (\bigoplus_i T^*S_i \otimes E_i) = 2nm$. □

Definition 3.2.4. For (finite-dimensional) complex vector spaces V, W, and $k \in \mathbb{N}$ denote by $j^k(V,W)$ the vector space of polynomial maps $f : V \to W$ of degree $\mathsf{deg}\, f \leqslant k$ with $f(0) = 0$, considered as the space of k-jets of holomorphic maps $F : V \to W$. For $l \geqslant k$ the natural projection $\mathsf{pr} : j^l(V,W) \to j^k(V,W)$ is well-defined. Let $j^{k,l}(V,W)$ denote its kernel. Similar notation for complex bundles is used. Note that $j^1(V,W) = \mathsf{Hom}\,(V,W) = V^* \otimes W$.

Lemma 3.2.3. *i) For any $(u, J_S, J; \boldsymbol{z}) \in \widehat{\mathscr{M}}_{\boldsymbol{k}}$ the jet $j^{2k_i+1}u(z_i^*)$ is a well-defined element of $j^{2k_i+1}(T_{z_i^*}, T_{u(z_i^*)}X) = j^{2k_i+1}(TS_i, E_i)_{(u,J_S,J;\boldsymbol{z})}$.*

ii) Moreover, $j^{2k_i+1}u(z_i^*) \in j^{k_i+1,2k_i+1}(T_{z_i^*}, T_{u(z_i^*)}X) = j^{k_i+1,2k_i+1}(TS_i, E_i)_{(u,J_S,J;\boldsymbol{z})}$.

iii) Set $\Upsilon_{\boldsymbol{k}}(u, J_S, J; \boldsymbol{z}) := \big(j^{k_1+1,2k_1+1}u(z_1^*), \dots, j^{k_m+1,2k_m+1}u(z_m^*)\big)$. Then $\Upsilon_{\boldsymbol{k}} : \widehat{\mathscr{M}}_{\boldsymbol{k}} \to \bigoplus_i j^{k_i+1,2k_i+1}(TS_i, E_i)$ is a section which is C^ℓ-smooth and transversal to the zero-section.

Proof. Assertions *i)* and *ii)* follow essentially from *Lemma 1.2.5*. The nontrivial points here are the following. First, the jet $j^{2k_i+1}u(z_i^*)$ is defined even if the structure J is C^ℓ-smooth with $\ell < 2k_i$ and the map u is $C^{\ell+1}$-smooth, since in general there are no higher smoothness for u. Second, the jet $j^{2k_i+1}u(z_i^*)$ is a *complex* polynomial. Finally, the jet

$j^{2k_i+1}u(z_i^*)$ is independent of the choice of the integrable structure J_{st} and J_{st}-holomorphic coordinates in a neighborhood of $u(z_i^*)$ used in *Lemma 1.2.5* for definition of the jet. Let us give a proof of the latter property.

Let J' and J'' be integrable complex structures in a neighborhood of $u(z_i^*)$ such that $J'(u(z_i^*)) = J''(u(z_i^*)) = J(u(z_i^*))$. Find local complex coordinate systems $\boldsymbol{w}' = (w_1', \dots, w_n')$ and $\boldsymbol{w}'' = (w_1'', \dots, w_n'')$ which are centered in $u(z_i^*)$ and holomorphic with respect to J' and J'' respectively. Without loss of generality we may assume that the frames $(\frac{\partial}{\partial w_1'}, \dots, \frac{\partial}{\partial w_n'})$ and $(\frac{\partial}{\partial w_1''}, \dots, \frac{\partial}{\partial w_n''})$ coincide in $u(z_i^*) \in X$. Consequently, we can express one system by another using the formula $\boldsymbol{w}'' = \boldsymbol{w}' + F(\boldsymbol{w}')$ with

$$F(\boldsymbol{w}') = O(|\boldsymbol{w}'|^2) \qquad \text{and} \qquad dF(\boldsymbol{w}') = O(|\boldsymbol{w}'|). \tag{3.2.3}$$

Let $u'(z)$ and $u'(z)$ be the local expressions of $u : S \to X$ in the local coordinate systems $\boldsymbol{w}'$ and $\boldsymbol{w}''$ respectively. Then $u''(z) = u'(z) + F(u'(z))$. So from (3.2.3) and $u'(z) = O(|z|^{k_i+2})$ we see that coefficients of polynomials $j^{2k_i+1}u'(z)$ and $j^{2k_i+1}u''(z)$ coincide.

To show the smoothness of the section $\Upsilon_{\boldsymbol{k}}$ we fix an element $(u_0, J_{S,0}, J_0; \boldsymbol{z}_0) \in \widehat{\mathscr{M}_{\boldsymbol{k}}}$, $\boldsymbol{z}_0 = (z_{1,0}^*, \dots, z_{m,0}^*)$, and a sufficiently small neighborhood $\mathscr{Y} \subset \widehat{\mathscr{M}_{\boldsymbol{k}}}$ of $y_0 := (u_0, J_{S,0}, J_0; \boldsymbol{z}_0)$. In what follows, for any $i = 1, \dots, m$, we fix families of certain structures on various spaces. We assume that the members of the families are parameterized by and depend C^ℓ-smoothly on $y = (u, J_S, J; \boldsymbol{z}) \in \mathscr{Y}$. The families are:

1. integrable complex structures J_i' in a neighborhood of each $u(z_{i,0}^*)$ such that each J_i' coincides with J in $u(z_i^*)$;
2. local complex coordinate systems $\boldsymbol{w}_i' = (w_{i,1}', \dots, w_{i,n}')$ on X centered in $u(z_i^*)$ and holomorphic with respect to J_i';
3. local frames $\boldsymbol{\xi}_i = (\xi_{i,1}, \dots, \xi_{i,n})$ of the bundles E_i which are defined in a neighborhood of z_i^* and holomorphic along S_i;
4. local J_S-holomorphic coordinates z_i on S_i centered in z_i^*.

Further, we assume that every coordinate z_i has image the whole disc Δ. Note that pulling back the frames $(\frac{\partial}{\partial w_{i,1}'}, \dots, \frac{\partial}{\partial w_{i,n}'})$ we obtain local frames $\Big(u^*(\frac{\partial}{\partial w_{i,1}'}), \dots, u^*(\frac{\partial}{\partial w_{i,n}'})\Big)$ of E_i which depend C^ℓ-smoothly on $y = (u, J_S, J; \boldsymbol{z}) \in \mathscr{Y}$. Now, the expression of $u(z)$ in the local coordinate system $\boldsymbol{w}_i'$ yields an element $u_i'(z_i) \in L^{1,p}(\Delta, \mathbb{C}^n)$ which depends C^ℓ-smoothly on $y \in \mathscr{Y}$ with respect to the standard smooth structure in $L^{1,p}(\Delta, \mathbb{C}^n)$. Deriving, we obtain an element $du_i'(z_i) \in L^p(\Delta, \mathbb{C}^n \otimes_{\mathbb{R}} T^*\Delta)$ which depends C^ℓ-smoothly on $y \in \mathscr{Y}$ with respect to the standard smooth structure in $L^p(\Delta, \mathbb{C}^n \otimes_{\mathbb{R}} T^*\Delta)$.

Consider now du_i' as a section of $E_i \otimes T^*S_i$, and its coefficients of du_i' in the frame $\Big(u^*(\frac{\partial}{\partial w_{i,1}'}) \otimes dz_i, \dots, u^*(\frac{\partial}{\partial w_{i,n}'}) \otimes dz_i\Big)$ as $L^p(\Delta, \mathbb{C})$-functions. Thus we can conclude that the coefficients of du_i' depend C^ℓ-smoothly on $y \in \mathscr{Y}$ with respect to the standard smooth structure in $L^p(\Delta, \mathbb{C})$. Consequently, the same is true for the coefficients of du_i' in the frame $(\xi_{i,1} \otimes dz_i, \dots, \xi_{i,n} \otimes dz_i)$. Since the latter frame is holomorphic, the coefficients of the jet $j^{2k_i}du(z_i^*)$ depend C^ℓ-smoothly on $y \in \mathscr{Y}$. This provides the desired smoothness property of $\Upsilon_{\boldsymbol{k}}$.

Finally, note that the transversality of $\Upsilon_{\boldsymbol{k}}$ to the zero-section follows from results of *Paragraph 3.1*. □

Corollary 3.2.4. *For any $\boldsymbol{k} = (k_1, \dots, k_m)$ with $k_i \geqslant 1$ the space $\widehat{\mathscr{M}_{\boldsymbol{k}}}$ is a C^ℓ-submanifold of $\widehat{\mathscr{M}}^{(m)}$ of codimension $2|\boldsymbol{k}|n$.*

Proof. Assume that for a given $\boldsymbol{k} = (k_1, \dots, k_m)$ with $k_i \geqslant 1$ the claim holds. Fix some $\boldsymbol{k}^+ = (k_1^+, \dots, k_m^+)$ with $k_i \leqslant k_i^+ \leqslant 2k_i$ and consider truncated section $\Upsilon_{\boldsymbol{k},\boldsymbol{k}^+} : \widehat{\mathscr{M}_{\boldsymbol{k}}} \to \bigoplus_i j^{k_i+1,k_i^++1}(TS_i, E_i)$ given by

$$\Upsilon_{\boldsymbol{k},\boldsymbol{k}^+}(u, J_S, J; \boldsymbol{z}) := \big(j^{k_1+1,k_1^++1}u(z_1^*), \dots, j^{k_m+1,k_m^++1}u(z_m^*)\big).$$

Then $\widehat{\mathscr{M}_{\boldsymbol{k}^+}}$ is identified with the zero set of $\Upsilon_{\boldsymbol{k},\boldsymbol{k}^+}$. By *Lemma 3.2.3*, $\Upsilon_{\boldsymbol{k},\boldsymbol{k}^+}$ is transversal to the zero-section. Thus $\widehat{\mathscr{M}_{\boldsymbol{k}^+}}$ is a C^ℓ-smooth submanifold of $\widehat{\mathscr{M}_{\boldsymbol{k}}}$ of codimension equal to $\mathsf{rank}_{\mathbb{R}} \bigoplus_i j^{k_i+1,k_i^++1}(TS_i, E_i) = 2n(|\boldsymbol{k}^+| - |\boldsymbol{k}|)$. So we can apply the induction. □

Lemma 3.2.5. *The natural projection $\widehat{\mathsf{pr}}_{\boldsymbol{k}} : \widehat{\mathscr{M}_{\boldsymbol{k}}} \to \widehat{\mathscr{M}}$ given by the formula $\widehat{\mathsf{pr}}_{\boldsymbol{k}}(u, J_S, J; \boldsymbol{z}) := (u, J_S, J)$ is an immersion of codimension $2(|\boldsymbol{k}|n - m)$.*

Proof. The differential of the projection $\widehat{\mathsf{pr}}_{\boldsymbol{k}}$ is given by

$$d\widehat{\mathsf{pr}}_{\boldsymbol{k}} : (v, \dot{J}_S, \dot{J}; \dot{\boldsymbol{z}}) \in T_{(u,J_S,J;\boldsymbol{z})}\widehat{\mathscr{M}_{\boldsymbol{k}}} \mapsto (v, \dot{J}_S, \dot{J}) \in T_{(u,J_S,J)}\widehat{\mathscr{M}}.$$

Thus the kernel $\mathsf{Ker}\, d\widehat{\mathsf{pr}}_{\boldsymbol{k}}$ consists of vectors of the form $(0,0,0;\dot{\boldsymbol{z}})$ with $\dot{\boldsymbol{z}}_i = (\dot{z}_1^*, \dots, \dot{z}_m^*) \in \bigoplus T_{z_i^*}S_i$ and we must show that $\mathsf{Ker}\, d\widehat{\mathsf{pr}}_{\boldsymbol{k}}$ is trivial. Intuitively this is obvious, since elements of the kernel correspond to deformations leaving (u, J_S, J) unchanged but moving cusp-points z_i^* on S and this is impossible.

For a rigorous proof we use conclusions of the proof of *Lemma 3.2.3*. Consider $du(z_i)$ as a holomorphic section of $T^*S_i \otimes E_i$. Then $du(z_i)$ vanishes in $z_{i,t}^*$ up to the order $\geqslant k_i$ and there are no other zeros of $du(z_i)$ in a neighborhood of z_i^*. Thus we can locally express z_i^* as the zero set of $du(z_i)$. This implies that locally there exists C^ℓ-smooth functions F_i of $(u, J_S, J) \in \widehat{\mathscr{M}}$ such that $F_i(u, J_S, J) = z_i^*$ for $(u, J_S, J) \in \widehat{\mathscr{M}_{\boldsymbol{k}}}$. Thus $\widehat{\mathsf{pr}}_{\boldsymbol{k}} : \widehat{\mathscr{M}_{\boldsymbol{k}}} \to \widehat{\mathscr{M}}$ is an immersion.

To compute the codimension of $\widehat{\mathsf{pr}}_{\boldsymbol{k}} : \widehat{\mathscr{M}_{\boldsymbol{k}}} \hookrightarrow \widehat{\mathscr{M}}$ one represents $\widehat{\mathsf{pr}}_{\boldsymbol{k}}$ as the composition $\widehat{\mathscr{M}_{\boldsymbol{k}}} \hookrightarrow \widehat{\mathscr{M}}^{(m)} \xrightarrow{\mathsf{pr}} \widehat{\mathscr{M}}$. □

Now we can finish

Proof of *Theorem 3.2.1*. Consider the action of $\mathbf{G}$ on $\widehat{\mathscr{M}}$ and the diagonal action of $\mathbf{G}$ on $\widehat{\mathscr{M}} \times (S)^m$. The both actions are C^ℓ-smooth, free, and commute with the projection $\mathsf{pr} : \widehat{\mathscr{M}} \times (S)^m \to \widehat{\mathscr{M}}$. Moreover, for every $\boldsymbol{k} = (k_1, \dots, k_m)$ with $k_i \geqslant 1$ the submanifold $\widehat{\mathscr{M}_{\boldsymbol{k}}} \hookrightarrow \widehat{\mathscr{M}} \times (S)^m$ is $\mathbf{G}$-invariant with respect to the diagonal action of $\mathbf{G}$. For the quotient $\mathscr{M}_{\boldsymbol{k}} = \widehat{\mathscr{M}_{\boldsymbol{k}}}/\mathbf{G}$ one can construct a C^ℓ-smooth atlas in the same way as it was done for $\mathscr{M} = \widehat{\mathscr{M}}/\mathbf{G}$. The construction shows that the map $\mathscr{M}_{\boldsymbol{k}} \to \mathscr{M}$ is a C^ℓ-smooth immersion of codimension equal to the codimension of $\widehat{\mathsf{pr}}_{\boldsymbol{k}} : \widehat{\mathscr{M}_{\boldsymbol{k}}} \hookrightarrow \widehat{\mathscr{M}}$. □

Summarizing the results and notation of this paragraph, we obtain

Corollary 3.2.6. *The maps $\mathsf{ev}_{\boldsymbol{k}} : \mathscr{M}_{\boldsymbol{k}} \to X^m$ and $\mathsf{ev}_i : \mathscr{M}_{\boldsymbol{k}} \to X$ given by $\mathsf{ev}_{\boldsymbol{k}}([u, J, \boldsymbol{z}]) := (u(z_1^*), \dots, u(z_m^*))$ and $\mathsf{ev}_i([u, J, \boldsymbol{z}]) := u(z_i^*)$ are well-defined and C^ℓ-smooth. This yields C^ℓ-smooth bundles $E_i := \mathsf{ev}_i^* TX$ with a fiber $(E_i)_{[u,J,\boldsymbol{z}]} = T_{u(z_i^*)}X$. The bundles $T_{z_i^*}S_i$ over $\widehat{\mathscr{M}_{\boldsymbol{k}}}$ induce C^ℓ-smooth bundles L_i over $\mathscr{M}_{\boldsymbol{k}}$ with the fiber $(L_i)_{[u,J,\boldsymbol{z}]} = T_{z_i^*}S_i$. The section $\Upsilon_{\boldsymbol{k}} : \widehat{\mathscr{M}_{\boldsymbol{k}}} \to \bigoplus_i j^{2k_i+1}(TS_i, E_i)$ induces the section $\Upsilon_{\boldsymbol{k}} : \mathscr{M}_{\boldsymbol{k}} \to \bigoplus_i j^{2k_i+1}(L_i, E_i)$ with $\Upsilon_{\boldsymbol{k}}([u, J, \boldsymbol{z}]) := \Upsilon_{\boldsymbol{k}}(u, J_S, J; \boldsymbol{z})$.*

Proof. The claim follows from the fact that all the constructions are compatible with $\mathbf{G}$-action. □

Definition 3.2.5. For a given $\boldsymbol{k}=(k_1,\dots,k_m)$ we set

$$\widehat{\mathscr{M}_{=\boldsymbol{k}}} := \big\{(u,J_S,J;\boldsymbol{z})\in\widehat{\mathscr{M}_{\boldsymbol{k}}}\;:\;\mathsf{ord}_{z_i^*}du=k_i\big\};\tag{3.2.4}$$

$$\mathscr{M}_{=\boldsymbol{k}} := \big\{[u,J;\boldsymbol{z}]\in\mathscr{M}_{\boldsymbol{k}}\;:\;\mathsf{ord}_{z_i^*}du=k_i\big\}.\tag{3.2.5}$$

Lemma 3.2.7. *i) The set $\widehat{\mathscr{M}_{=\boldsymbol{k}}}$ is an open $C^{\ell-1}$-smooth submanifold of $\widehat{\mathscr{M}_{\boldsymbol{k}}}$ invariant with respect to the natural action of $\mathbf{G}$ on $\widehat{\mathscr{M}_{\boldsymbol{k}}}$.*

ii) The image of the projection of $\widehat{\mathscr{M}_{=\boldsymbol{k}}}$ to $\widehat{\mathscr{M}}$ is an imbedded submanifold of $\widehat{\mathscr{M}}$, and the projection is a non-ramified covering over the image.

iii) There exists a $C^{\ell-1}$-smooth bundle N over $\widehat{\mathscr{M}_{=\boldsymbol{k}}}\times S$ whose restriction onto $\{(u,J_S,J;\boldsymbol{z})\}\times S$ coincides with N_u. The diagonal action of $\mathbf{G}$ on $\widehat{\mathscr{M}_{=\boldsymbol{k}}}\times S$ lifts canonically to the action on the bundle N.

iv) The bundle N induces Banach bundles $L^{1,p}(S,N)$ and $L^p_{(0,1)}(S,N)$ over $\widehat{\mathscr{M}_{=\boldsymbol{k}}}$ with fibers $L^{1,p}(S,N_u)$ and $L^p_{(0,1)}(S,N_u)$ over $(u,J_S,J,\boldsymbol{z})\in\widehat{\mathscr{M}_{=\boldsymbol{k}}}$ respectively. The operators $D^N_{u,J}:L^{1,p}(S,N_u)\to L^p_{(0,1)}(S,N_u)$ induce a $C^{\ell-1}$-smooth bundle homomorphism $D^N: L^{1,p}(S,N)\to L^p_{(0,1)}(S,N)$.

Proof. *i)* The complement $\widehat{\mathscr{M}_{\boldsymbol{k}}}\backslash\widehat{\mathscr{M}_{=\boldsymbol{k}}}$ is of the union of (the projections of) the spaces $\widehat{\mathscr{M}_{\boldsymbol{k}'}}$ such that either $\boldsymbol{k}'=(k_1,\dots,k_m,1)$, or $\boldsymbol{k}'=(k'_1,\dots,k'_m)$ with $k'_i\geqslant k_i$ and $k'_{i_0}>k_{i_0}$ for some i_0. In other words, we have either at least one additional cusp-point or a higher order cusp in at least one point. Obviously, these conditions define closed subsets in $\widehat{\mathscr{M}_{\boldsymbol{k}}}$. The $\mathbf{G}$-invariance of $\widehat{\mathscr{M}_{=\boldsymbol{k}}}$ follows from the definition.

ii) The set $\widehat{\mathscr{M}_{=\boldsymbol{k}}}$ admits a finite transformation group $\mathsf{Aut}(\boldsymbol{k})$ generated by transpositions of marked cusp-points z_i^* and z_j^* with $k_i=k_j$. The rest of part *ii)* follows.

iii) Let z_i be a local J_S-holomorphic coordinate on $\widehat{\mathscr{M}_{=\boldsymbol{k}}}\times S$ centered at z_i^* as in *Definition 3.2.3*. It follows from the proof of *Lemma 3.2.3* that $z_i^{-k_i}du(z_i)$ is a well-defined *non-vanishing* local section of $\mathsf{Hom}\,(TS,E_u)$, which depends $C^{\ell-1}$-smoothly on (u,J_S,J) and holomorphically on z_i. This provides the existence on N with the stated property, at least locally in a neighborhood of $(u,J_S,J;z_i^*)$. The globalization of N is trivial. Since the constructions involved are natural, the $\mathbf{G}$-action admits the desired lift.

iv) One uses the fact that the constructions of the bundles $L^{1,p}(S,N)$, $L^p_{(0,1)}(S,N)$, and the operator D^N are natural. This implies $C^{\ell-1}$-smoothness of the obtained objects. □

Remark. One could explain the meaning of *Lemma 3.2.7* as follows. First, we note that for the globalization of normal bundles N_u to $\mathscr{M}$ we should use not the Cartesian product $\mathscr{M}\times S$, but the $\mathbf{G}$-twisted product $\mathscr{M}\ltimes S$, *i.e.* $\widehat{\mathscr{M}}\times_{\mathbf{G}}S:=\big(\widehat{\mathscr{M}}\times S\big)/\mathbf{G}$. Second, we must choose a stratification of $\mathscr{M}$ by strata where N_u does not “jump”. By the definition of N_u such strata are exactly $\mathscr{M}_{=\boldsymbol{k}}=\widehat{\mathscr{M}_{=\boldsymbol{k}}}/\mathbf{G}$ where there is no “jump” of the cusp-order.

Another application of the techniques used in the proof of *Theorem 3.2.1* is a local version of the theorem. Below $\mathscr{P}(\Delta,X)$ denotes the Banach space of pseudoholomorphic maps between the unit disc Δ with the standard structure J_{st} and X, *i.e.* $\mathscr{P}(\Delta,X)=\{(u,J)\in L^{1,p}(\Delta,X)\times\mathscr{J}:\overline{\partial}_{J_{\mathsf{st}},J}u=0\}$.

Lemma 3.2.8. *i) For any given integer $k\geqslant 1$ the set*

$$\mathscr{P}_k(\Delta,0;X):=\{(u,J)\in\mathscr{P}(\Delta,X):\mathsf{ord}_{z=0}(du)\geqslant k\}\tag{3.2.6}$$

is a C^ℓ*-smooth submanifold of* $\mathscr{P}(\Delta, X)$ *of real codimension* $2kn$*,* $n := \mathsf{dim}_{\mathbb{C}} X = \frac{1}{2}\mathsf{dim}_{\mathbb{R}} X$*, with tangent space*

$$T_{(u,J)}\mathscr{P}_k(\Delta, 0; X) = \{(v, \dot{J}) \in T_u L^{1,p}(\Delta, X) \times T_J \mathscr{J} : D_{u,J} v = 0, j^k(v(z) - v(0)) = 0\}. \tag{3.2.7}$$

3.3. **Curves with prescribed secondary cusp index.** Recall that by *Lemma 1.2.4* for a pseudoholomorphic map $u : (S, J_S) \to (X, J)$ with cusp order k at $z^* \in S$ the jet $j^{2k+1}u(z^*)$ is well-defined. As we shall see, the part of the jet $j^{2k+1}u(z^*)$ invariant under reparameterization plays an important role for determining the type of critical points on moduli spaces (see *Paragraph 4.3*).

Definition 3.3.1. *i)* Let $u : (S, J_S) \to (X, J)$ be pseudoholomorphic map with a cusp of order $k := \mathsf{ord}_{z^*} du$ at $z^* \in S$, $\mathsf{pr}_N : E_u \to N_u$ the projection to the normal bundle, and z a local holomorphic coordinate on S centered at z^*. Define the *secondary cusp index* l *of* u *at* $z^* \in S$ by setting $l := k$ if $\mathsf{pr}_N \circ j^{2k+1}u(z^*)$ is zero polynomial and $l := \mathsf{ord}_{z=0}\mathsf{pr}_N \circ j^{2k+1}u(z^*) - k - 1$ otherwise.

ii) For a given m-tuple $\boldsymbol{k} = (k_1, \ldots, k_m)$ of prescribed orders of cusps we consider m-tuples $\boldsymbol{l} = (l_1, \ldots, l_m)$ with $0 \leqslant l_i \leqslant k_i$ and set $|\boldsymbol{l}| := \sum_i l_i$. Define the *moduli space* $\mathscr{M}_{\boldsymbol{k},\boldsymbol{l}}$ *of pseudoholomorphic maps with cusps of given order and secondary index* $(\boldsymbol{k}, \boldsymbol{l})$ as the set of $[u, J, \boldsymbol{z}] \in \mathscr{M}_{\boldsymbol{k}}$ such that $\mathsf{ord}_{z_i^*} du = k_i$ and the secondary cusp index of u at z_i^* is at least l_i. Set

$$\widehat{\mathscr{M}_{\boldsymbol{k},\boldsymbol{l}}} := \{(u, J_S, J, \boldsymbol{z}) \in \widehat{\mathscr{M}_{=\boldsymbol{k}}} : [u, J, \boldsymbol{z}] \in \mathscr{M}_{\boldsymbol{k},\boldsymbol{l}}\}. \tag{3.3.1}$$

Theorem 3.3.1. *The space* $\mathscr{M}_{\boldsymbol{k},\boldsymbol{l}}$ *is a closed* $C^{\ell-1}$*-smooth submanifold of* $\mathscr{M}_{=\boldsymbol{k}}$ *of codimension* $2(n-1)|\boldsymbol{l}|$*.*

Remark. The meaning of the notion of secondary cusp index can be explained as follows. One expects that for a J-holomorphic map $u : S \to X$ with a cusp of order $k = \mathsf{ord}_{z^*} du$ at $z^* \in S$ the polynomial $\mathsf{pr}_N \circ j^{2k+1}u(z^*)$ has vanishing order $k+1$. Thus the secondary cusp index l is the order of deviation from this condition. The content of *Theorem 3.3.1* is that for generic $[u, J; \boldsymbol{z}] \in \mathscr{M}_{=\boldsymbol{k}}$ there is no deviation and that the space of curves with cusps of prescribed degeneration order is of expected codimension.

We note also that the range $0 \leqslant l_i \leqslant k_i = \mathsf{ord}_{z_i^*} du$ is the maximal one where the secondary cusp index is well-defined: The higher order terms of $\mathsf{pr}_N \circ du$, as well as the coefficients of du (considered as a holomorphic section of $T^*S \otimes E_u$), depend on the choice of the local holomorphic coordinate z_i centered at $z_i^* \in S$.

Proof. We maintain the notation of *Lemma 3.2.3*. Now, for any $(u, J_S, J; \boldsymbol{z}) \in \widehat{\mathscr{M}_{=\boldsymbol{k}}}$, $\boldsymbol{z} = (z_1^*, \ldots, z_m^*)$, the jets $j^{2k_i+1}u(z_i^*) \in j^{2k_i+1}(TS_i, E_i)_{(u,J_S,J;\boldsymbol{z})}$ are well-defined and depend $C^{\ell-1}$-smoothly on $(u, J; \boldsymbol{z})$. By *Lemma 3.2.7*, for any $i = 1, \ldots, m$ the formula $(N_i)_{(u,J_S,J;\boldsymbol{z})} := (N_u)_{z_i^*}$ defines a $C^{\ell-1}$-smooth bundle N_i over $\widehat{\mathscr{M}_{=\boldsymbol{k}}}$ with the projection $\mathsf{pr}_N : E_i \to N_i$. This yields the compositions $\mathsf{pr}_N \circ j^{2k_i+1}u(z_i^*) \in j^{2k_i+1}(TS_i, N_i)_{(u,J_S,J;\boldsymbol{z})}$ which depend $C^{\ell-1}$-smoothly on $(u, J_S, J; \boldsymbol{z})$. Thus we obtain a $C^{\ell-1}$-smooth bundle

$$\bigoplus_{i=1}^m j^{k_i+1, k_i+l_i+1}(TS_i, N_i)_{(u,J_S,J;\boldsymbol{z})}$$

over $\widehat{\mathscr{M}_{=\boldsymbol{k}}}$ of rank $2(n-1)|\boldsymbol{l}|$ and a $C^{\ell-1}$-smooth section

$$\Upsilon^N_{\boldsymbol{k},\boldsymbol{l}} := (\mathsf{pr}_N \circ j^{k_i+1, k_i+l_i+1} u(z_i^*))_{i=1}^m.$$

Observe that $\widehat{\mathscr{M}_{\boldsymbol{k},\boldsymbol{l}}}$ is defined in $\widehat{\mathscr{M}_{=\boldsymbol{k}}}$ as the zero set of $\Upsilon^N_{\boldsymbol{k},\boldsymbol{l}}$. It follows from *Lemma 3.2.3* that $\Upsilon^N_{\boldsymbol{k},\boldsymbol{l}}$ is transversal to the zero section. Consequently, $\widehat{\mathscr{M}_{\boldsymbol{k},\boldsymbol{l}}}$ is a submanifold of $\widehat{\mathscr{M}_{=\boldsymbol{k}}}$ of codimension $2(n-1)|\boldsymbol{l}|$. The claim of the theorem follows now by taking the $\mathbf{G}$-quotient. $\square$

3.4. **Curves with cusps of prescribed type.** In this paragraph we give a construction of J-curves of any given cusp type, completing the result of Micallef and White. In particular, we obtain a more direct and constructive proof of *Lemma 1.2.1* without referring to local structure of minimal surfaces, as is done in [Mi-Wh]. Then we show that the set of cusp-curves with prescribed cusp type is a Banach submanifold of the total moduli space and compute its codimension.

Let J be an almost complex structure on the ball $B\subset\mathbb{C}^n$ such that $J(0)=J_{\mathsf{st}}(0)$. We assume that J is C^ℓ-smooth with $\ell\geqslant 2$. First we consider the local structure of multiple maps.

Lemma 3.4.1. *Let $u:\Delta\to B$ be a non-constant J-holomorphic map with $u(0)=0\in B$. Then there exist a radius $r>0$, a uniquely defined $\nu\in\mathbb{N}$, and a non-multiple J-holomorphic map $u':\Delta(r^\nu)\to B$ such that $u(z)=u'(z^\nu)$ for $z\in\Delta(r)$.*

Proof. By *Lemma 1.2.5*, $u(z)=v\cdot z^\mu+O(|z|^{\mu+\alpha})$ with some $v\in T_0B=\mathbb{C}^n$, positive $\mu\in\mathbb{N}$, and $\alpha>0$. If $\mu=1$, then u is already non-multiple in some $\Delta(r)$ and there are nothing to prove. Thus we may assume that $\mu\geqslant 2$.

Take a sufficiently small $\rho_0>0$ and consider $U:=u^{-1}(B(\rho_0))$. By the first part of *Lemma 1.2.5*, U is a disc and u is an immersion in $U\backslash\{0\}$. Using the second part of *Lemma 1.2.5* it is not difficult to show that $u(U\backslash\{0\})$ is an immersed J-holomorphic punctured disc in B. Therefore the restriction $u_{|U}$ is a composition of a non-multiple J-holomorphic map and a covering branched only in $0\in U$. $\square$

It is known that any non-multiple holomorphic map $u:\Delta\to\mathbb{C}^n$, in appropriate holomorphic coordinates on Δ and $\mathbb{C}^n$, has locally the form

$$u(z)=\sum_{i=0}^{l} v_i z^{p_i},$$

with the following properties. $p_0=\mathsf{ord}_0(du)+1$, $v_0\neq 0$, the vectors $v_i\in\mathbb{C}^n$ are linearly independent of v_0 for $i>0$, and $\mathsf{gcd}(p_0,\dots,p_l)=1$. We want to establish a similar result for pseudoholomorphic curves, replacing the operation $u_{i-1}(z)\mapsto u_{i-1}(z)+v_iz^{p_i}$ by $u_{i-1}(z)\mapsto \mathsf{dfrm}_{p_i}(u_{i-1},v_i)$.

Lemma 3.4.2. *Let $B\subset\mathbb{C}^n$ be the unit ball, J a C^ℓ-smooth almost complex structure on B with $J(0)=J_{\mathsf{st}}$, and $u:\Delta\to B$ a non-multiple J-holomorphic map such that $u(z)=v_0z^{p_0}+o(z^{p_0})$ for some $p_0>1$ and $v_0\neq 0\in\mathbb{C}^n$. Take a divisor $d>1$ of p_0 and denote by η a primitive d-th root of unity. Then there exist an integer $q>0$, a vector $v\in\mathbb{C}^n$, and a complex polynomial $\psi(z)$ such that*

i) q is not a multiple of d;

ii) v is $\mathbb{C}$-linearly independent of v_0; in particular, $v\neq 0$;

iii) $\psi(z)=z+o(z)$ and $\mathsf{deg}\,\psi(z)\leqslant q$;

iv) $u(\eta z)=u(\psi(z))+z^q\cdot v+o(z^q)$. (3.4.1)

Proof. Denote by $v_0^\perp \subset \mathbb{C}^n$ a complex orthogonal complement to v_0, and by $B^\perp(\rho)$ the ball of radius ρ in $v_0^\perp$. Note that we can canonically identify the space $v_0^\perp$ with the fiber $(N_u)_{z=0}$ of the normal bundle N_u of u (see *Definition 1.5.1*). Fixing a holomorphic frame $w_1(z), \dots w_{n-1}(z)$ of N_u we can identify $\Delta \times v_0^\perp$ with the total space of N_u over Δ and use $(z, w_1, \dots w_{n-1})$ as coordinates in $\Delta \times v_0^\perp$. Denote by J_{st} the standard integrable complex structure in $\Delta \times v_0^\perp$. It coincides with the canonical holomorphic structure in N_u. Set

$$U_{r,\rho} := \Delta(r) \times B^\perp(\rho)$$

Fix a holomorphic splitting $F_0 : N_u \to E_u$ of the projection $\mathsf{pr}_N : E_u \to N_u$. We shall identify N_u as a subbundle of E_u by means of F_0. Define the map $F : U_{r,\rho} \to \mathbb{C}^n$ as the composition

$$(z,w) \mapsto F_0(z)(w) \in (E_u)_z = T_{u(z)}B = \mathbb{C}^n \mapsto F(z,w) := u(z) + F_0(z)(w).$$

It is not difficult to see that for sufficiently small r and ρ the map $F = F(z,w)$ takes values in B and has the following properties:

- $F(z,w)$ is C^1-smooth;
- $F(z,0) = u(z)$ and $\nabla_{\dot w} F(z,0) = \dot w$; or more precisely, $\nabla_{\dot w} F(z,0) = F_0(z)(\dot w)$;
- the pulled-back structure $\tilde J := F^* J$ coincides with J_{st} along the set $\check\Delta \times \{0\}$, *i.e.*

$$\tilde J(z,0) = J_{\mathsf{st}}(z,0). \tag{3.4.2}$$

From (3.4.2) we obtain a uniform estimate

$$\big| \tilde J(z,w) - J_{\mathsf{st}}(z,w) \big| \leqslant C \cdot |w|. \tag{3.4.3}$$

Further, $\eta^{p_0} = 1$ obviously gives $u(\eta z) - u(z) = o(z^{p_0}) = o(z^d)$. This implies that for sufficiently small r' we can represent $u(z)$ in the form $u(z) = F(\zeta(z), \tilde w(z))$ with uniquely defined C^1-smooth $\zeta(z) : \Delta(r') \to \Delta(r)$ and $\tilde w(z) : \Delta(r') \to B^\perp(\rho)$ fulfilling the condition $\zeta(z) = z + o(z)$. Further, $\tilde w(z) = o(z^d)$.

Set $\tilde u(z) := \big(\zeta(z), \tilde w(z)\big)$. We obtain a C^1-smooth map $\tilde u(z) : \Delta(r') \to U_{r,\rho}$, for which

$$|J_{\mathsf{st}}(\tilde u(z)) - \tilde J(\tilde u(z))| = |J_{\mathsf{st}}(\zeta(z), \tilde w(z)) - \tilde J(\zeta(z), \tilde w(z))| \leqslant C' \cdot |\tilde w(z)|.$$

Consequently

$$\big|\overline\partial_{J_{\mathsf{st}}} \tilde u(z)\big| = \big|\overline\partial_{J_{\mathsf{st}}} \tilde u(z) - \overline\partial_{\tilde J} \tilde u(z)\big| = \big|\big(J_{\mathsf{st}}(\tilde u(z)) - \tilde J(\tilde u(z))\big)\partial_y \tilde u(z)\big| \leqslant C'' \cdot |\tilde w(z)|,$$

or explicitly for components $\zeta(z)$ and $\tilde w(z)$

$$\big|\overline\partial_{J_{\mathsf{st}}} \tilde w(z)\big| \leqslant C'' \cdot |\tilde w(z)|; \tag{3.4.4}$$

$$\big|\overline\partial_{J_{\mathsf{st}}} \zeta(z)\big| \leqslant C'' \cdot |\tilde w(z)|. \tag{3.4.5}$$

Observe that $\tilde w(z)$ is not identically zero. Otherwise we would obtain that $u(\eta z) = u(\zeta(z))$, which would contradict the condition of non-multiplicity of $u(z)$.

Hence, by *Lemma 1.2.1*, $\tilde w(z) = z^q v + o(z^q)$ and $\zeta(z) = \psi(z) + o(z^q)$ for some $q > 0$, non-zero $v \in v_0^\perp$, and a complex polynomial $\psi(z)$ of degree $\leqslant q$. Substituting these relations in $u(z) = F(\tilde u(z))$ we obtain (3.4.1).

Finally, the identity $\sum_{j=1}^d \big(u(\eta^j z) - u(\eta^{j-1} z)\big) \equiv 0$ together with (3.4.1) implies that $\sum_{j=1}^d (\eta^{j-1} z)^q \cdot v = 0$. Thus $\sum_{j=1}^d \eta^{jq} = 0$ which is possible if and only if q is not a multiple of d. □

Iterating the construction of *Lemma 3.4.2*, we obtain

Corollary 3.4.3. *Let $B \subset \mathbb{C}^n$ be the unit ball, J a C^ℓ-smooth almost complex structure in B with $J(0) = J_{\mathsf{st}}$, and $u : \Delta \to B$ a non-multiple J-holomorphic map with $u(0) = 0$.*

Then there exist uniquely defined sequences $(p_0, p_1, \dots, p_l)$ and $(d_0, d_1, \dots, d_l)$ of positive integers with the following properties:

i) $p_0 = \mathsf{ord}_0 du + 1$, so that $u(z) = z^{p_0} v_0 + o(z^{p_0})$ with non-zero $v_0 \in \mathbb{C}^n$;

ii) $d_i = \mathsf{gcd}(p_0, \dots, p_i)$;

iii) $p_i < p_{i+1}$, $d_i > d_{i+1}$, and $d_l = 1$; in particular, p_{i+1} is not a multiple of d_i; (3.4.6)

iv) if η_i is the primitive d_i-th root of unity, then

$$u(\eta_i z) = u(\psi_i(z)) \cdot v_0 + z^{p_{i+1}} \cdot v_{i+1} + o(z^{p_{i+1}})$$

for appropriate complex polynomials $\psi_i(z)$ with $\psi_i(z) = z + o(z)$, and vector $v_{i+1} \in \mathbb{C}^n$, $\mathbb{C}$-linearly independent of v_0.

Definition 3.4.1. *i)* To any increasing sequence of positive integers $1 \leqslant p_0 < p_1 < \dots < p_l$ we associate the *sequence of divisors* $d_i \geqslant d_1 \geqslant \dots \geqslant d_l$ defined by $d_i = \mathsf{gcd}(p_0, \dots, p_i)$. In particular, $d_0 = p_0$.

ii) A sequence $\vec{p} = (p_0, p_1, \dots, p_l)$ of positive integer exponents is called a *cusp type* if p_i and the associate divisors $d_i = \mathsf{gcd}(p_0, \dots, p_i)$ satisfy the condition (3.4.6). In the situation of *Corollary 3.4.3*, the sequence $\vec{p} = (p_0, p_1, \dots, p_l)$ is called the *cusp type of u at $z = 0$*, p_i the *critical exponents of u at $z = 0$*, and $\vec{d} = (d_i)$ the *sequence of divisors of u at $z = 0$*.

iii) For a given cusp type $\vec{p} = (p_0, p_1, \dots, p_l)$, an integer p' is called an *admissible exponent* if p' equals p_l or is of the form $p' = p_i + j \cdot d_i$ for some $i = 0, \dots, l-1$ and $j = 0, \dots, l_i$, $l_i := \left[\frac{p_{i+1} - p_i}{d_i}\right]$. Thus all critical exponents are admissible and there are exactly l_i non-critical admissible exponents between p_i and p_{i+1}. Denote by $\vec{p}' = (p'_0, \dots, p'_{l'})$ the sequence of the admissible exponents ordered by growth. Its length is $l' = l + \sum_{i=0}^{l-1} l_i = l + \sum_{i=0}^{l-1} \left[\frac{p_{i+1} - p_i}{d_i}\right]$.

Note that the corresponding sequence of divisors $d'_j := \mathsf{gcd}(p'_0, \dots, p'_j)$ consists of divisors d_i of critical exponents, repeated $l_i + 1$ times. Vice versa, an admissible exponent $p'_j > p'_0 = p_0$ is critical if and only if $d'_j < d'_{j-1}$.

Theorem 3.4.4. *Let $B \subset \mathbb{C}^n$ be the unit ball, J an almost complex on B with $J(0) = J_{\mathsf{st}}$, and $u : \Delta \to B$ a non-multiple J-holomorphic map such that $u(0) = 0$. Further, let $\vec{p} = (p_0, \dots, p_l)$ and $\vec{p}' = (p_0, \dots, p_{l'})$ be the sequences of critical and resp. admissible exponents of u at $z = 0$, and $\vec{d}' = (d'_0, \dots, d'_{l'})$ the corresponding sequence of divisors.*

Then there exist a sequence $(v_0, \dots, v_{l'})$ of vectors in $\mathbb{C}^n$ (one v_j for each p'_j), a complex polynomial $\varphi(z)$, and a radius $r > 0$, such that the following holds.

i) $u(z) = z^{p_0} \cdot v_0 + o(z^{p_0})$; $v_0 \neq 0$, $v_1, \dots, v_{l'}$ are complex orthogonal to v_0; (3.4.7)

ii) $\varphi(z) = z + o(z)$ and $\deg \varphi(z) \leqslant p_l - p_0 + 1$; (3.4.8)

iii) for appropriately chosen maps dfrm, the recursive formula

$$u_j(z) := \mathsf{dfrm}_{p'_j/d'_j}\big(u_{j-1}(z^{d'_{j-1}/d'_j}), J; v_j\big) \quad j = 0, 1, \dots, l' \tag{3.4.9}$$

beginning from $u_{-1}(z) \equiv 0$ yields a sequence of well-defined J-holomorphic maps $u_j : \Delta(r^{d'_j}) \to B$ with the property

$$u(\varphi(z)) - u_j(z^{d'_j}) = v_{j+1} z^{p'_{j+1}} + o(z^{p'_{j+1}}). \tag{3.4.10}$$

Moreover, such v_j and $\varphi(z)$ are uniquely defined. Further, v_j is non-zero if p'_j is critical.

Proof. The choice of the maps dfrm_d ensuring that at each recursive step the right hand side of (3.4.10) is well-defined will be made below. Now we assume that for given $j < l'$ we have constructed a J-holomorphic map $u_j : \Delta(r^{d'_j}) \to X$ and a polynomial $\varphi(z)$ such that $\varphi(z) = z + o(z)$ and $u(\varphi(z)) = u_j(z^{d'_j}) + o(z^{p'_j})$. Then by *Lemma 1.2.5*,

$$u(\varphi(z)) = u_j(z^{d'_j}) + z^q w + o(z^q) \tag{3.4.11}$$

for some non-zero $w \in \mathbb{C}^n$ and $q > p'_j$. Represent $w \in \mathbb{C}^n$ in the form $w = a \cdot v_0 + w'$ and replace φ by $\varphi'(z) := \varphi(z) - \frac{a}{p_0} \cdot z^{q-p_0+1}$. The relations $u(z) = z^{p_0} v_0 + o(z^{p_0})$ and (3.4.11) yield

$$u(\varphi'(z)) = u_j(z^{d'_j}) + z^q w' + o(z^q). \tag{3.4.12}$$

If $w' = 0$, we can consider (3.4.11) with some $q' > q$. Thus we may assume that $w' \neq 0$. Moreover, we see that $\varphi(z)$ is defined uniquely by (3.4.10) up to degree $p'_j - p_0 + 1$.

Denote by η_j the primitive d'_j-th root of unity. Then by *Lemma 3.4.2*,

$$u(\eta_j z) = u(\psi_j(z)) + v z^p + o(z^p) \tag{3.4.13}$$

for an appropriate polynomial $\psi_j(z) = z + o(z)$ and v linearly independent of v_0. Moreover, p is the first critical exponent after p'_j in the sequence $\vec{p}'$ of the admissible exponents of $u(z)$ at $z = 0$. In particular, p is not a multiple of d'_j. Set $\hat\varphi_{j+1}(z) := \eta_j^{-1} \varphi_{j+1}(\eta_j z)$. Then we obtain $\hat\varphi_{j+1}(z) = z + o(z)$ and

$$\begin{aligned} u(\varphi_{j+1}(\eta_j z)) &= u(\eta_j \hat\varphi_{j+1}(z)) = u(\psi_j(\hat\varphi_{j+1}(z))) + z^p v + o(z^p) \\ &= u(\varphi_{j+1}(\hat\psi_j(z))) + z^p v + o(z^p), \end{aligned} \tag{3.4.14}$$

where $\hat\psi_j(z)$ is a polynomial with $\hat\psi_j(z) = z + o(z)$ and $\hat\psi_j(\hat\varphi_{j+1}(z)) = \varphi_{j+1}(\hat\psi_j(z)) + o(z^p)$. Substitution of (3.4.12) in (3.4.14) together with the identity $\eta_j^{d'_j} = 1$ yields

$$u_j(z^{d'_j}) + \eta_j^q z^q w' = u_j\big(\hat\psi_j^{d'_j}(z)\big) + z^q w' + v z^p + o(z^{\mathsf{min}(p,q)}). \tag{3.4.15}$$

Further, since $\hat\psi_j(z) = z + o(z)$, we can find a polynomial $\tilde\psi_j(z)$ with the properties $\tilde\psi_j(z) = z + o(z)$ and $\hat\psi_j^{d'_j}(z) = \tilde\psi_j(z^{d'_j})$. For such $\tilde\psi_j(z)$, the relation (3.4.15) transforms to

$$u_j(z^{d'_j}) = u_j\big(\tilde\psi_j(z^{d'_j})\big) + (1 - \eta_j^q) z^q w' + v z^p + o(z^{\mathsf{min}(p,q)}). \tag{3.4.16}$$

Assume that $q < p$. Then q is a multiple of d'_j. In particular, $q \geqslant p'_{j+1}$. In the case $q > p'_{j+1}$ we simply set $v_{j+1} := 0$ and obtain the relation (3.4.10). In the case $q = p'_{j+1}$ we set $v_{j+1} := w'$ and come to the relation (3.4.10) again. The case $p < q$ is impossible since p is not a multiple of d'_j.

In the remaining case $q = p$ we have two subcases, $p'_{j+1} < p$ and $p'_{j+1} = p$. Then we set $v_{j+1} := 0$ or respectively $v_{j+1} := w'$ and obtain (3.4.10) from (3.4.12).

Now we construct the maps $\mathsf{dfrm}_{p'_j/d_j}$ with the desired properties. The idea is to rescale the maps u_j making the norms $\|du_j\|_{L^2}$ sufficiently small and obtaining a recursive apriori estimate on v_j. For this fix some $r \in\,]0,1[$ and maps $\widetilde{\mathsf{dfrm}}_p$ with the properties listed in *Definition 3.1.1*. Then the substitutions $\tilde u(z) := u(rz)$, $\tilde u_j(z) := u_j(r^{d'_j} z)$, $\tilde v_j := r^{p'_j} v_j$, and $\tilde\varphi_j(z) := r^{-1}\varphi_j(rz)$ transform (3.4.9) and (3.4.10) into recursive relations

$$\tilde u_j(z) = \widetilde{\mathsf{dfrm}}_{p'_j/d'_j}\big(\tilde u_{j-1}(z^{d'_{j-1}/d'_j}), J; \tilde v_j\big), \tag{3.4.17}$$

$$\tilde u(\tilde\varphi(z)) - \tilde u_j(z^{d'_j}) = \tilde v_{j+1} z^{p'_{j+1}} + o(z^{p'_{j+1}}). \tag{3.4.18}$$

for J-holomorphic maps $\tilde u_j:\Delta\to B$. Note that $\|d\tilde u\|_{L^2(\Delta)}=\|du\|_{L^2(\Delta(r))}$ will be arbitrarily small for r small enough. Choosing an appropriate $r\ll 1$ and using induction, on can obtain sufficiently small upper bounds on $\tilde v_j$, ensuring that (3.4.17) is well-defined for $j=0,\dots,l$. For such r, we define $\mathsf{dfrm}_{p'_j/d_j}$ by the reverse substitutions in (3.4.17). □

Remark. For almost complex surface, *i.e.* in the case $n=2$, the critical exponents determine a topological type of a cusp. In particular, under hypotheses of *Theorem 3.4.4*, the intersection of the image $u(\Delta)$ with the sphere S^3_r of a sufficiently small radius $r>0$ is an iterated toric knot γ transversal to the 2-plane distribution ξ on S^3_r given by $\xi_x := T_xS^3_r\cap J(x)T_xS^3_r$. Thus the Bennequin index $\beta(\gamma,\xi)$ is well-defined. We refer to [Iv-Sh-1] for the proof of the formula $\beta=2\delta-1$ relating the Bennequin index β of γ and the nodal number δ of $u(\Delta)$ in $0\in B$. On the other hand, δ can be computed by the formula

$$\delta=\sum_{i=1}^{m}(d_{i-1}-d_i)(p_i-1), \tag{3.4.19}$$

see [Rf] or [Mil]. In the higher dimensional setting, *i.e.* for $n\geqslant 3$, the topological type of the cusp $u(\Delta)$ is not determined by the critical exponents and depends on additional information encoding further linear relations between v_j. For example, the condition *v_2 and v_1 are linearly dependent* defines a proper subset in $\mathscr{P}_{\vec p}(\Delta,0;B)$. Moreover, using the techniques of this paragraph one can show that this subset is a $C^{\ell-1}$-smooth submanifold in $\mathscr{P}_{\vec p}(\Delta,0;B)$. Details can be recovered by an interested reader.

Theorem 3.4.5. *Let $B\subset\mathbb{C}^n$ be the unit ball, $\vec p=(p_0,\dots,p_l)$ a cusp type, $\vec p'=(p_0,\dots,p_{l'})$ the corresponding sequences of admissible exponents, and $\vec d'=(d'_0,\dots,d'_{l'})$ the sequence of divisors associated with $\vec p'$. Then the set*

$$\mathscr{P}_{\vec p}(\Delta,0;B):=\big\{(u,J)\in\mathscr{P}_{p_0-1}(\Delta,0;B): u\text{ has a cusp type }\vec p\text{ in }z=0\,\big\} \tag{3.4.20}$$

is a $C^{\ell-1}$-smooth submanifold of $\mathscr{P}_{p_0-1}(\Delta,0;B)$ of real codimension $2(n-1)(p_l-p_0-l')$.

Note that $p_0=p'_0$ and $p_l=p'_{l'}$.

Proof. Let $u:\Delta(r)\to B$ be a J-holomorphic map, $r>0$, and let

$$\mathscr{P}_k(\Delta,u;B,J):=\{u'\in\mathscr{P}(\Delta;B,J): u(z)-u'(z)=o(z^k)\}$$

By *Theorem 3.1.3*, $\mathscr{P}_k(\Delta,u;B,J)$ is a $C^{\ell-1}$-smooth submanifold of $\mathscr{P}(\Delta;B,J)$ of codimension $2n(k+1)$. For $l>k$ it follows that $\mathscr{P}_l(\Delta,u;B,J)$ has codimension $2n(l-k)$ in $\mathscr{P}_k(\Delta,u;B,J)$. Moreover, if J_y is a C^ℓ-smooth family of almost complex structures in B parameterized by a (Banach) manifold $\mathscr{Y}$ and $u_y\in L^{1,p}(\Delta(r),B)$ a $C^{\ell-1}$-smooth family of J_y-holomorphic maps, then $\cup_{y\in\mathscr{Y}}\mathscr{P}_k(\Delta,u_y;B,J_y)$ is a $C^{\ell-1}$-smooth manifold.

For a given $(u^*,J^*)\in\mathscr{P}_{\vec p}(\Delta,0;B)$, let $v^*_0,\dots,v^*_{l'}$ and $\varphi^*(z)=z+\varphi^*_2z^2+\varphi^*_3z^3+\cdots$ be the parameters of u^* constructed in *Theorem 3.4.4*. Define $\mathscr{Y}$ to be the space of small deformations of v^*_j and φ^*_i. This means that $y\in\mathscr{Y}$ is a tuple $(v_0,\dots,v_{l'};\varphi_2,\dots,\varphi_{p_l-p_0+1})$ with $v_j\in\mathbb{C}^n$ and $\varphi_i\in\mathbb{C}$ satisfying $|v_j-v^*_j|<\varepsilon$ and $|\varphi_i-\varphi^*_i|<\varepsilon$ with ε sufficiently small. Further, let U denote a sufficiently small neighborhood of J^* in the space of C^ℓ-smooth almost complex structures in B. For $y=(v_0,\dots;\varphi_2,\dots)\in\mathscr{Y}$ and $J\in U$ we construct the maps $u_{y,J;j}$, $j=0,\dots,l'$, using the recursive relation (3.4.9) and set $\varphi_y(z):=z+\varphi_2z^2+\cdots+\varphi_{p_l-p_0+1}z^{p_l-p_0+1}$. Then for $|z|<r'\ll 1$ the inverse map $\varphi_y^{-1}(z)$ is well-defined and holomorphic. Define $u_{y,J}(z):=u_{y,J;l'}(\varphi^{-1}(z))$. We obtain a $C^{\ell-1}$-smooth family of pseudoholomorphic maps $u_{y,J}:\Delta(r')\to B$ parameterized by $(y,J)\in\mathscr{Y}\times U$.

Note that by *Theorem 3.4.4* every $(u,J) \in \mathscr{P}_{\vec{p}}(\Delta,0;B)$ sufficiently close to (u^*,J^*) lies in $\mathscr{P}_{p_l}(\Delta,u_{y,J};B,J)$ for an appropriate $y \in \mathscr{Y}$, and such $y \in \mathscr{Y}$ is uniquely defined. Thus the union $\cup_{(y,J)\in\mathscr{Y}\times U}\mathscr{P}_{p_l}(\Delta,u_{y,J};B,J)$ is a local $C^{\ell-1}$-smooth chart for $\mathscr{P}_{\vec{p}}(\Delta,0;B)$.

Finally, note that the union $\cup_{(y,J)\in\mathscr{Y}\times U}\mathscr{P}_{p_0-1}(\Delta,u_{y,J};B,J)$ is naturally isomorphic to $\mathscr{P}_{p_0}(\Delta,0;B)\times\mathscr{Y}$. Computing the number of parameters we obtain the codimension of the imbedding $\mathscr{P}_{\vec{p}}(\Delta,0;B)\hookrightarrow\mathscr{P}_{p_0}(\Delta,0;B)$. □

Globalizing *Theorem 3.4.5* we obtain

Corollary 3.4.6. *Let $\vec{\boldsymbol{p}} = (\vec{p}_1,\ldots,\vec{p}_m)$ be a sequence of cusp types, $\vec{p}_i = (p_{i,0},\ldots p_{i,l_i})$. Set $k_i := p_{i,0}-1$ and $\boldsymbol{k} := (k_1,\ldots,k_m)$. Then the space*

$$\mathscr{M}_{\vec{\boldsymbol{p}}} := \{[u,J;z_1^*,\ldots z_m^*] \in \mathscr{M}_{\boldsymbol{k}} : u \text{ has cusp type } \vec{p}_i \text{ in } z_i^*\ \}$$

is a $C^{\ell-1}$-smooth submanifold of $\mathscr{M}_{\boldsymbol{k}}$ of codimension $2(n-1)\sum_{i=0}^{m}(p_{i,l_i}-p_{i,0}-l'_i)$.

4. Saddle points in the moduli space

4.1. **Critical and saddle points in the moduli space.** In application of the continuity method for constructing J-holomorphic curves two main difficulties occur. The first one appears in the proof of the "closedness part", when one tries to extend a deformation $[u_t,J_t]\in\mathscr{M}, t\in[0,t'[$ into the endpoint t'. This difficulty is connected with the fact that the projection $\pi_{\mathscr{J}}:\mathscr{M}\to\mathscr{J}$ is, in general, not proper. In particular, for a path $J_t\in\mathscr{J}, t\in[0,t']$ there may not exist a lift $[u_t,J_t]$ to $\mathscr{M}$, and the fibers $\mathscr{M}_{\mathscr{J}}=\pi_J^{-1}(J)$ can be non-compact. Gromov's compactness theorem ([Gro], see also [Iv-Sh-3]) gives a fiberwise topological compactification of $\mathscr{M}$ by adding certain degenerate curves C. However, for the moment we neglect this difficulty assuming we can avoid it in our case.

The second main difficulty appears in the proof of the "openness part" when one tries to extend a lift $[u_t,J_t]\in\mathscr{M}$, $t\in[0,t']$, of a path of $J_t\in\mathscr{J}, t\in[0,1]$ to a bigger interval $t\in[0,t''[$ with some $t''>t'$. Obviously, this difficulty can appear only if $[u_{t'},J_{t'}]$ is a *critical point* of $\pi_{\mathscr{J}}$, *i.e.* when the differential of the projection $d\pi_{\mathscr{J}}$ is not surjective in $[u_{t'},J_{t'}]$. Thus it is desirable to find conditions on the critical points of $\pi_{\mathscr{J}}$ which ensure the existence of such a lift.

Assume additionally that the given path $h:[0,1]\to\mathscr{J}$, $h(t):=J_t$, is C^2-smooth and transversal to $\pi_h:\mathscr{M}\to\mathscr{J}$, *i.e.* $\mathscr{M}_h$ is a manifold. Let $\pi_h:\mathscr{M}_h\to I$, $I:=[0,1]$, be the projection. Then we have a well-defined C^1-smooth bundle homomorphism $d\pi_h:\mathscr{M}_h\to\pi_h^*(TI)\cong\mathbb{R}$. Further, $d\pi_h$ vanishes exactly at critical points of π_h and, by *Lemma 1.3.1*, at each such point $p:=[u,h(t)]$ we have a well-defined quadratic form $\nabla d\pi_h(p):T_p\mathscr{M}_h\to\mathbb{R}\cong T_tI$. For our purpose it is sufficient to show that each critical point p is a *saddle*, *i.e.* the quadratic form $\nabla d\pi_h(p)$ has at least one positive and one negative eigenvalue.

It turns out that this condition depends only on the geometry of the projection $\pi_{\mathscr{J}}:\mathscr{M}\to\mathscr{J}$ at $p=[u,h(t)]$, and not on the particular choice of a transversal map $h:I\to\mathscr{J}$. In more detail, the situation is as follows.

First, since $\mathscr{M}$ is C^ℓ-smooth with $\ell\geqslant 2$, the map $\pi_{\mathscr{J}}:\mathscr{M}\to\mathscr{J}$ defines a C^1-smooth homomorphism of Banach bundles $d\pi_{\mathscr{J}}:T\mathscr{M}\to\pi_{\mathscr{J}}^*(T\mathscr{J})$. *Corollary 2.2.3* relates the (co)kernel of $d\pi_{\mathscr{J}}$ for a given $[u,J]\in\mathscr{M}$ with $\mathsf{H}^i(S,\mathscr{N}_u)$, and *Lemma 1.3.1* provides a well-defined bilinear map

$$\Phi=\Phi_{[u,J]}:=\nabla d\pi_{\mathscr{J}}:T_{[u,J]}\mathscr{M}\times\mathsf{H}^0(S,\mathscr{N}_u)\to\mathsf{H}^1(S,\mathscr{N}_u). \tag{4.1.1}$$

The situation remains essentially the same if we consider a relative moduli space $\mathscr{M}_h = Y \times_h \mathscr{M}$ with a C^ℓ-smooth map $h : Y \to \mathscr{J}$ transversal to $\pi_{\mathscr{J}}$. Indeed, one can easily see that for the natural projection $\pi_h : \mathscr{M}_h \to Y$ and a point $[u, y] \in \mathscr{M}_h$ with $h(y) =: J$ one has the natural isomorphisms

$$\begin{array}{rcll} \mathsf{Ker}\,(d\pi_h : T_{[u,y]}\mathscr{M}_h \to T_yY) & \cong & \mathsf{Ker}\,(d\pi_{\mathscr{J}} : T_{[u,J]}\mathscr{M} \to T_J\,\mathscr{J}\,) & \cong \mathsf{H}^0_D(S, \mathscr{N}_u), \\ \mathsf{Coker}\,(d\pi_h : T_{[u,y]}\mathscr{M}_h \to T_yY) & \cong & \mathsf{Coker}\,(d\pi_{\mathscr{J}} : T_{[u,J]}\mathscr{M} \to T_J\,\mathscr{J}\,) & \cong \mathsf{H}^1(S, \mathscr{N}_u). \end{array} \tag{4.1.2}$$

Further, the relation between $\Phi = \nabla d\pi_{\mathscr{J}}$ and $\nabla d\pi_h$ is given by the following

Lemma 4.1.1. *i) The isomorphism* $\mathsf{Coker}\,(d\pi_h) \cong \mathsf{H}^1(S, \mathscr{N}_u)$ *is induced by composition* $T_yY \xrightarrow{dh} T_J\,\mathscr{J} \xrightarrow{\overline{\Psi}_{u,J}} \mathsf{H}^1(S, \mathscr{N}_u)$.

ii) The bilinear map $\nabla d\pi_h : T_{[u,y]}\mathscr{M}_h \times \mathsf{H}^0_D(S, \mathscr{N}_u) \to \mathsf{H}^1(S, \mathscr{N}_u)$ *is induced by the composition* $T_{[u,y]}\mathscr{M}_h \hookrightarrow T_{[u,J]}\mathscr{M} \oplus T_yY \twoheadrightarrow T_{[u,J]}\mathscr{M}$ *and the bilinear map* $\Phi : T_{[u,J]}\mathscr{M} \times \mathsf{H}^0_D(S, \mathscr{N}_u) \to \mathsf{H}^1(S, \mathscr{N}_u)$.

Summing up, we obtain the following situation in the most important case $Y = I$.

Lemma 4.1.2. *For a map* $h : I \to \mathscr{J}$ *transversal to* π_J*, the singular points of the projection* $\pi_h : \mathscr{M}_h \to I$ *are exactly those* $[u, t] \in \mathscr{M}_h$ *for which* $\mathsf{H}^1(S, \mathscr{N}_u) = \mathbb{R}$.

For such $[u, t] \in \mathscr{M}_h$ *with* $J := h(t)$ *one has the equality* $T_{[u,t]}\mathscr{M}_h = \mathsf{H}^0_D(S, \mathscr{N}_u)$ *and the isomorphism* $\overline{\Psi}_{[u,J]} : T_tI \xrightarrow{\cong} \mathsf{H}^1(S, \mathscr{N}_u)$*. Moreover, the quadratic form* $\Phi_{[u,J]} : \mathsf{H}^0_D(S, \mathscr{N}_u) \to \mathsf{H}^1(S, \mathscr{N}_u)$ *equals to the composition of the quadratic form* $\nabla d\pi_h : T_{[u,t]}\mathscr{M}_h \to T_tI$ *with* $\overline{\Psi}_{[u,J]} : T_tI \to \mathsf{H}^1(S, \mathscr{N}_u)$.

Corollary 4.1.3. *The nullity, rank and signature of* $\Phi_{[u,J]}$ *and* $\nabla d\pi_h$ *coincide.*

Definition 4.1.1. Let Q be a quadratic form defined on a (finite-dimensional) vector space V and taking values in a vector space W with $\mathsf{dim}_{\mathbb{R}} W = 1$. Define the *saddle index of* Q by $\mathsf{S\text{-}ind}\, Q := \min\{\mathsf{ind}_+ Q, \mathsf{ind}_- Q\}$, where $\mathsf{ind}_\pm Q$ are respectively the positive and negative indices of Q with respect to some (in fact, any) orientation of W. For a critical point $[u, J] \in \mathscr{M}$ with $\mathsf{H}^1(S, \mathscr{N}_u) \cong \mathbb{R}$, call $\mathsf{S\text{-}ind}\, \Phi_{[u,J]}$ the *saddle index of* $[u, J]$. A point $[u, J] \in \mathscr{M}$ is a *saddle point* of the moduli space $\mathscr{M}$ if and only if $\mathsf{S\text{-}ind}\, \Phi_{[u,J]}$ is strictly positive.

4.2. **Second variation of the $\bar\partial$-equation.** To find saddle points of $\mathscr{M}$ we need to find an explicit formula for the form Φ in (4.1.1). Note that, since the space $\mathscr{P}$ appears as the zero-set of the $\bar\partial$-equation (1.1.1), the description of the tangent space $T\mathscr{P}$ is given by the variation of the $\bar\partial$-equation. Similarly, we show that the form Φ is essentially the part of the second variation of the $\bar\partial$-equation invariant with respect to the choice of a connection being used.

Let $[u, J] \in \mathscr{M}$ be represented by $(u, J_S, J) \in \widehat{\mathscr{M}}$. Recall the description of $T_{(u,J_S,J)}\widehat{\mathscr{M}}$ given in (2.2.7). Moreover, since du is non-vanishing at a generic point, $\dot J_S$ is determined by v and $\dot J$. Note that the tangent space to an orbit $\mathbf{G} \cdot (u, J) \subset \widehat{\mathscr{M}}$ can be identified with $du(\mathsf{H}^0(S, TS)) \subset \mathscr{E}_{(u,J_S,J)}$. This defines a subbundle of $\mathscr{E}$ which we also denote by $du(\mathsf{H}^0(S, TS))$. Thus we obtain the isomorphism

$$T_{[u,J]}\mathscr{M} \cong T_{(u,J_S,J)}\widehat{\mathscr{M}}/du(\mathsf{H}^0(S, TS)). \tag{4.2.1}$$

Explicitly, the tangent space $T_{[u,J]}\mathscr{M}$ consists of triples $([v], \dot J_S, \dot J)$ for which $(v, \dot J_S, \dot J) \in T_{(u,J_S,J)}\widehat{\mathscr{M}}$ and $[v]$ is the equivalence class $v + du\big(\mathsf{H}^0(S, TS)\big)$.

Definition 4.2.1. Set

$$\widehat{\mathscr{E}}_{(u,J_S,J)} := \big(\mathscr{E}_{(u,J_S,J)}/du(\mathsf{H}^0(S,TS)\big) \oplus \mathsf{H}^1(S,TS). \tag{4.2.2}$$

Recall the canonical isomorphism $T_{J_S}\mathbb{T}_g \cong \mathsf{H}^1(S,TS)$ given by (2.2.4). For $(u,J_S,J) \in \widehat{\mathscr{M}}$ define the operator

$$\widehat{D} = \widehat{D}_{u,J} : \widehat{\mathscr{E}}_{(u,J_S,J)} \to \mathscr{E}'_{(u,J_S,J)} \qquad \widehat{D}([v],[I_S]) := Dv + J \circ du \circ I_S. \tag{4.2.3}$$

Lemma 4.2.1. *Formula* (4.2.2) *defines a C^ℓ-smooth Banach bundle $\widehat{\mathscr{E}}$ over $\mathscr{M}$ with the fiber $\widehat{\mathscr{E}}_{(u,J_S,J)}$ over $[u,J] \in \mathscr{M}$. The tangent bundle $T\mathscr{M}$ can be included in the following exact sequence of bundles over $\mathscr{M}$*

$$0 \to T\mathscr{M} \xrightarrow{\alpha} \widehat{\mathscr{E}} \oplus \pi^*_{\mathscr{J}} T\mathscr{J} \xrightarrow{\beta} \mathscr{E}' \to 0, \tag{4.2.4}$$

where the homomorphisms $\alpha = (\alpha_1,\alpha_2)$ and $\beta = (\beta_1,\beta_2)$ are given by

$$\begin{array}{rcl} \alpha_1([v],\dot{J}_S,\dot{J}) & := & ([v],[\dot{J}_S]) \in \widehat{\mathscr{E}} = \mathscr{E}/du\big(\mathsf{H}^0(S,TS)\big) \bigoplus \mathsf{H}^1(S,TS) \\ \alpha_2 & := & d\pi_{\mathscr{J}} : T_{[u,J]}\mathscr{M} \to T_J\mathscr{J} \\ \beta_1 & := & \widehat{D}_{u,J} : \widehat{\mathscr{E}}_{(u,J_S,J)} \to \mathscr{E}'_{(u,J_S,J)} \\ \beta_2 & := & \Psi_{u,J} : T_J\mathscr{J} \to \mathscr{E}'_{(u,J_S,J)} \end{array} \tag{4.2.5}$$

Proof. It is easy to show that $\mathsf{H}^0(S,TS)$ and $\mathsf{H}^1(S,TS)$ can be considered as smooth bundles over $\widehat{\mathscr{M}}$ equipped with the natural $\mathbf{G}$-action. Then du defines a $\mathbf{G}$-equivariant homomorphism between the bundles $\mathsf{H}^0(S,TS)$ and $\mathscr{E}$. Hence, using formula (4.2.2), we can construct a bundle $\widehat{\mathscr{E}}_{\widehat{\mathscr{M}}}$ over $\widehat{\mathscr{M}}$ with the induced $\mathbf{G}$-action. By *Lemma 2.2.2 i)*, this is equivalent to the first assertion of the lemma.

The exactness of (4.2.4) follows from relations (2.2.7) and (4.2.1–4.2.3). □

Lemma 4.2.2. *The homomorphisms α_1 and β_2 yield isomorphisms*

$$\mathsf{H}^0_D(S,\mathscr{N}_u) \cong \mathsf{Ker}\,\alpha_2 \overset{\alpha_1}{\cong} \mathsf{Ker}\,\beta_1 \quad \textit{and} \quad \mathsf{H}^1_D(S,\mathscr{N}_u) \cong \mathsf{Coker}\,\alpha_2 \overset{\beta_2}{\cong} \mathsf{Coker}\,\beta_1, \tag{4.2.6}$$

inducing the identity

$$\Phi_{u,J} = -\nabla\widehat{D} : T_{[u,J]}\mathscr{M} \times \mathsf{H}^0_D(S,\mathscr{N}_u) \to \mathsf{H}^1_D(S,\mathscr{N}_u). \tag{4.2.7}$$

Proof. The isomorphisms (4.2.6) follow from definitions and *Corollary 2.2.3*. Moreover, we can identify $\mathsf{H}^0_D(S,\mathscr{N}_u)$ with $\mathsf{Ker}\big(\widehat{D}_{u,J} : \widehat{\mathscr{E}}_{(u,J_S,J)} \to \mathscr{E}'_{(u,J_S,J)}\big)$.

Let $i : \mathsf{H}^0_D(S,\mathscr{N}_u) \to T_{[u,J]}\mathscr{M}$ and $p : \mathscr{E}'_{(u,J_S,J)} \to \mathsf{H}^1_D(S,\mathscr{N}_u)$ denote the corresponding inclusion and projection. Fix some connections on $T\mathscr{M}$, $\pi^*_{\mathscr{J}}T\mathscr{J}$, $\widehat{\mathscr{E}}$, and $\mathscr{E}'$, and denote all of them simply by ∇. Covariant differentiation of the relation $\beta_1 \circ \alpha_1 + \beta_2 \circ \alpha_2 = 0$ gives

$$\nabla\beta_1 \circ \alpha_1 + \nabla\beta_2 \circ \alpha_2 + \beta_1 \circ \nabla\alpha_1 + \beta_2 \circ \nabla\alpha_2 = 0, \tag{4.2.8}$$

which together with $\alpha_2 \circ i = 0$ and $p \circ \beta_1 = 0$ yields

$$p \circ \nabla\beta_1 \circ \alpha_1 \circ i = -p \circ \beta_2 \circ \nabla\alpha_2 \circ i. \tag{4.2.9}$$

□

Definition 4.2.2. Using the isomorphisms $\mathsf{Ker}\big(\widehat{D}_{u,J} : \widehat{\mathscr{E}}_{(u,J_S,J)} \to \mathscr{E}'_{(u,J_S,J)}\big) \cong \mathsf{H}^0_D(S,\mathscr{N}_u)$ from (4.2.6) and $\mathsf{H}^1(S,TS) \cong T_{J_S}\mathbb{T}_g$ from (2.2.4), redefine

$$\mathsf{H}^0_D(S,\mathscr{N}_u) := \big\{\, ([v],I_S) \in \widehat{\mathscr{E}}_{(u,J_S,J)} \oplus T_{J_S}\mathbb{T}_g \;:\; Dv + J\circ du\circ I_S = 0 \,\big\}. \tag{4.2.10}$$

Then the projection $\mathsf{H}^0_D(S,\mathscr{N}_u) \to \mathsf{H}^0_D(S,N_u)$ is given by the formula $([v],I_S) \mapsto \mathsf{pr}_N(v)$ with $\mathsf{pr}_N : E_u \to N_u$ defined by (1.5.2).

Now assume that some symmetric connections on TX and TS are fixed. They induce connections on $\mathscr{E}$ and $\mathscr{E}'$, on the tangent bundles $T\widehat{\mathscr{M}}$ and $T\mathscr{M}$, and so on. We shall use the same notation ∇ for all these connections. Further, denote by $R^X(\cdot,\cdot;\cdot)$ the curvature operator of the connection ∇ on X.

Lemma 4.2.3. *For $([v],\dot J_S,\dot J) \in T_{[u,J]}\mathscr{M}$ and $([w],I_S) \in \mathsf{H}^0_D(S,\mathscr{N}_u) \subset \widehat{\mathscr{E}}_{(u,J_S,J)}$*

$$\begin{aligned}\big(\nabla_{([v],\dot J_S,\dot J)}\widehat{D}\big)([w],I_S) &= \underline{R^X(v,du;w)}_{[1]} + \underline{J\circ R^X(v,du\circ J_S;w)}_{[2]} + \\ &+ \underline{\nabla_v J\circ\nabla w\circ J_S}_{[3]} + \underline{\nabla^2_{v,w}J\circ du\circ J_S}_{[4]} + \underline{\nabla_w J\circ\nabla v\circ J_S}_{[5]} + \underline{\dot J\circ\nabla w\circ J_S}_{[6]} + \\ &+ \underline{\nabla_w\dot J\circ du\circ J_S}_{[7]} + \underline{J\circ\nabla w\circ\dot J_S}_{[8]} + \underline{\nabla_w J\circ du\circ\dot J_S}_{[9]} + \underline{\nabla_v J\circ du\circ I_S}_{[10]} + \\ &+ \underline{J\circ\nabla v\circ I_S}_{[11]} + \underline{\dot J\circ du\circ I_S}_{[12]}.\end{aligned} \tag{4.2.11}$$

Remark. The numerical subscripts on the various terms are for future reference.

Proof. Consider the bundle $\widetilde{\mathscr{E}} := \mathscr{E}\oplus\mathsf{H}^1(S,TS)$ over $\widehat{\mathscr{M}}$ with the bundle homomorphism $\widetilde{D} : \widetilde{\mathscr{E}} \to \mathscr{E}'$, $\widetilde{D}(w,[I_S]) := Dw + J\circ du\circ I_S$. We claim that for the covariant derivative $\big(\nabla_{([v],\dot J_S,\dot J)}\widetilde{D}\big)(w,I_S)$ we obtain the same expression as in the statement of the lemma. Obviously, this would imply the lemma.

The only nontrivial point here is to compute the derivative of the operator of covariant differentiation $\nabla^{\mathsf{op}} := \nabla : L^{1,p}(S,u^*TX) \to L^p(S,u^*TX\otimes T^*S)$ in the direction given by some $v \in T_uL^{1,p}(S,X) = L^{1,p}(S,u^*TX)$. To do this, we fix a smooth vector field ξ on S and a local section $\boldsymbol{w}$ of $\mathscr{E}$. Then $\big(\nabla^{\mathsf{op}}\boldsymbol{w}\big)(\xi) = \nabla_\xi\boldsymbol{w}$ is a local section of a Banach bundle with the fiber $L^p(S,u^*TX)$ over $u \in L^{1,p}(S,X)$.

Differentiation in the direction v yields

$$\nabla_v\big(\nabla^{\mathsf{op}}\boldsymbol{w}\big)(\xi) = \nabla^2_{v,\xi}\boldsymbol{w} = R^X(v,du(\xi);\boldsymbol{w}) + \nabla^2_{\xi,v}\boldsymbol{w} = \tag{4.2.12}$$

$$R^X(v,du;\boldsymbol{w})(\xi) + \big(\nabla^{\mathsf{op}}_\xi(\nabla\boldsymbol{w})\big)(v). \tag{4.2.13}$$

Thus we obtain the formula $\nabla_v(\nabla^{\mathsf{op}}) = R^X(v,du;\cdot)$. Besides, we have the relation $\nabla_v du = \nabla v$, which was already used for deriving (1.3.8) from (1.1.1). Now, the proof of the lemma can be completed by explicit calculations. □

Using (4.2.11) we can describe in more detail the structure of $\pi_{\mathscr{J}} : \mathscr{M} \to \mathscr{J}$ at critical points with $\mathsf{H}^1_D(S,N_u) \cong \mathbb{R}$. Note that the term [4] in (4.2.11) is the only one that depends on second order derivatives of J. Further, the operator $D = D_{u,J}$ is also independent of second order derivatives. Thus, deforming J and preserving the order one jet $j^1J|_{u(S)}$, the map $u : S \to X$ remains J-holomorphic with same the D-cohomology groups $\mathsf{H}^i(S,\mathscr{N}_u)$. The result of such changes of J is given by

Lemma 4.2.4. *Let $[u,J] \in \mathscr{M}$ with $\mathsf{H}^1_D(S,N_u) \cong \mathbb{R}$ and a quadratic form $\tilde\Phi : \mathsf{H}^0_D(S,N_u) \to \mathsf{H}^1_D(S,N_u)$ be given. Then there exists a C^1-small perturbation $\tilde J \in \mathscr{J}$ of J such that $j^1J|_{u(S)} = j^1\tilde J|_{u(S)}$ and the restriction of $\Phi_{u,\tilde J}$ to $\mathsf{H}^0_D(S,N_u)$ equals the given $\tilde\Phi$. Moreover,*

such a perturbation $\tilde J$ of J can be realized in an arbitrarily small neighborhood of a given point $x \in u(S)$.

Proof. Let $U \subset X$ be a neighborhood of the given x. Find $U' \subset U$ such that $u^{-1}(U') \neq \varnothing$ and u is an imbedding on $u^{-1}(U')$. Obviously, it is sufficient to find an appropriate jet $j^2\tilde J|_{u(S)}$ which differs from $j^2 J|_{u(S)}$ only in $U' \cap u(S)$. Then $j^2\tilde J|_{u(S)}$ can be extended to $\tilde J$ with the desired properties.

Covariant differentiation of the identity $J^2 = -\mathsf{Id}$ gives the relations $\nabla_v J \circ J + J \circ \nabla_v J = 0$ and

$$\nabla^2_{v_1,v_2} J \circ J + \nabla_{v_1} J \circ \nabla_{v_2} J + \nabla_{v_2} J \circ \nabla_{v_1} J + J \circ \nabla^2_{v_1,v_2} J = 0, \qquad v_1, v_2 \in T_x X. \tag{4.2.14}$$

Consequently, we have the following description of the possible choice for $j^2\tilde J|_{u(S)}$ with $j^1\tilde J|_{u(S)} = j^1 J|_{u(S)}$. The tensor field $u(S) \ni x \mapsto \Theta_x$ defined by

$$v_1, v_2, w \in T_x X \mapsto \Theta_x(v_1, v_2; w) := \nabla^2_{v_1,v_2}(\tilde J - J)(w) \in T_x X \tag{4.2.15}$$

must be supported in U', symmetric[2] in v_1 and v_2, J-antilinear in w, and zero for $v_1, v_2 \in T_x(u(S)) \subset T_x X$. Vice versa, any tensor field Θ with these properties has the form $\Theta(v_1, v_2; w) = \nabla^2_{v_1,v_2}(\tilde J - J)(w)$ for an appropriate $\tilde J$ with $j^1\tilde J|_{u(S)} = j^1 J|_{u(S)}$.

The condition that $\Theta_x(v_1, v_2; w)$ vanishes for $v_1, v_2 \in T_x(u(S))$ means that for $x \in U' \cap u(S)$ we can consider $\Theta_x(v_1, v_2; w)$ as a tensor with arguments v_1, v_2 varying in the normal bundle N_u. Now, for $\tilde J$ as above, $v \in \mathsf{H}^0_D(S, N_u)$, and $\psi \in \mathsf{H}^0_D(S, N_u \otimes K_S) \cong \mathsf{H}^1(S, N_u)^*$ we obtain the relation

$$\langle \psi, \Phi_{u,\tilde J}(v,v) \rangle = \langle \psi, \Phi_{u,J}(v,v) \rangle + \mathsf{Re} \int_S \psi \circ \Theta(v, v; du). \tag{4.2.16}$$

Finally, observe that any quadratic form on a *finite dimensional* space $\mathsf{H}^0_D(S, N_u)$ can be realized as $\mathsf{Re} \int_S \psi \circ \Theta(v, v; du)$ with Θ satisfying the conditions stated above. □

4.3. **Second variation at cusp-curves.** Our aim in this paragraph is to find conditions ensuring that a critical point $[u, J] \in \mathscr{M}$ with $\mathsf{H}^1(S, N_u) \cong \mathbb{R}$ is a saddle point. *Lemma 4.2.4* shows that such critical points with $\mathscr{N}_u^{\mathsf{sing}} \cong 0$ are "hopeless" from this point of view. Hence, we need to understand in more detail the structure of the bilinear operator Φ on the component $\mathsf{H}^0(S, \mathscr{N}_u^{\mathsf{sing}}) \subset \mathsf{H}^0_D(S, \mathscr{N}_u)$.

Recall that by the definition of the normal sheaf the stalk $(\mathscr{N}_u^{\mathsf{sing}})_z$ at $z \in S$ is non-trivial exactly when z is a cusp-point of $u : S \to X$ and in this case $\mathsf{dim}_{\mathbb{C}}(\mathscr{N}_u^{\mathsf{sing}})_z = \mathsf{ord}_z du$. Thus we want to understand the structure of the moduli space at critical points corresponding to cusp-curves. The following two lemmas contain technical results needed for this purpose. Recall that the holomorphic line bundle $\mathscr{O}([A])$ was introduced in *Definition 1.5.1*. We maintain the notation ∇ and $R^X(\cdot,\cdot;\cdot)$ from *Lemma 4.2.3*. In particular, we have $\nabla_\xi J = \nabla_{du(\xi)} J$, $\nabla^2_{\xi,\eta} v - \nabla^2_{\eta,\xi} v = R^X(du(\xi), du(\eta); v)$, and other similar relations. Further, we assume that $\nabla J_S = 0$.

Lemma 4.3.1. *An element $([w], I_S) \in \mathsf{H}^0_D(S, \mathscr{N}_u)$ lies in $\mathsf{H}^0(S, \mathscr{N}_u^{\mathsf{sing}})$ if and only if $w = du(\tilde w)$ for some $\tilde w \in L^{1,p}_{\mathsf{loc}}(S \backslash \mathsf{supp}\,(\mathscr{N}_u^{\mathsf{sing}}), TS)$ that extends to $\tilde w \in L^{1,p}(S, TS \otimes \mathscr{O}([A]))$.*

In this case, outside the zero-set of du one has

$$\bar\partial \tilde w \equiv \nabla \tilde w + J_S \circ \nabla \tilde w \circ J_S = -J_S \circ I_S \tag{4.3.1}$$

[2] Obviously, $\nabla^2_{v_1,v_2}(\tilde J - J) - \nabla^2_{v_2,v_1}(\tilde J - J)$ can be expressed via $\tilde J - J$ and the curvature tensor $R^X(\cdot,\cdot;\cdot)$ of ∇. But $\tilde J - J$ vanishes on $u(S)$.

and

$$
\begin{aligned}
0 \equiv{}& \nabla_{\tilde w}(D_{u,J}v + \dot J\circ du\circ J_S + J\circ du\circ \dot J_S) = \underline{D_{u,J}(\nabla_{\tilde w} v)}_{[D]} + \qquad (4.3.2)\\
&\underline{R^X(w,du;v)}_{[1']} + \underline{J\circ R^X(w,du\circ J_S;v)}_{[2']} + \underline{\nabla_w J\circ\nabla v\circ J_S}_{[5]} + \underline{\nabla^2_{w,v}J\circ du\circ J_S}_{[4']} +\\
&\underline{\nabla_v J\circ\nabla_{\tilde w}du\circ J_S}_{[3']} + \underline{\dot J\circ\nabla_{\tilde w}du\circ J_S}_{[6']} + \underline{\nabla_w\dot J\circ du\circ J_S}_{[7]} + \underline{J\circ\nabla_{\tilde w}du\circ\dot J_S}_{[8']} +\\
&\underline{\nabla_w J\circ du\circ\dot J_S}_{[9]} + \underline{J\circ du\circ\nabla_{\tilde w}\dot J_S}_{[13]} - \underline{\nabla v\circ\nabla\tilde w}_{[14]} - \underline{J\circ\nabla v\circ\nabla\tilde w\circ J_S}_{[15]}.
\end{aligned}
$$

Proof. Relation (4.3.1) follows from the equality

$$
\begin{aligned}
0 &= D_{u,J}(du(\tilde w)) + J\circ du\circ I_S = du(\overline\partial\tilde w) + du\circ J_S\circ I_S =\\
&= du\Big((\nabla\tilde w + J_S\circ\nabla\tilde w\circ J_S) + J_S\circ I_S\Big).
\end{aligned}
$$

Using $J_S^2 = -\mathsf{Id}$ we can write the relation in the form $I_S = J_S\circ\nabla\tilde w - \nabla\tilde w\circ J_S$.

To show (4.3.2) we start with the computation of $D_{u,J}(\nabla_{\tilde w}v)$:

$$
\begin{aligned}
&D_{u,J}(\nabla_{\tilde w}v) = \nabla(\nabla_{\tilde w}v) + J\circ\nabla(\nabla_w v)\circ J_S + \nabla J(\nabla_w v, du\circ J_S) = \qquad (4.3.3)\\
&\underline{\nabla^2_{\cdot,\tilde w}v}_{[16]} + \underline{\nabla v\circ\nabla\tilde w}_{[14]} + \underline{J\circ\nabla^2_{\cdot,\tilde w}v\circ J_S}_{[17]} + \underline{J\circ\nabla v\circ\nabla\tilde w\circ J_S}_{[15]} + \underline{\nabla J(\nabla_{\tilde w}v, du\circ J_S)}_{[18]}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\nabla_{\tilde w}(D_{u,J}v + \dot J\circ du\circ J_S + J\circ du\circ\dot J_S) = \qquad (4.3.4)\\
&\nabla_{\tilde w}(\nabla v + J\circ\nabla v\circ J_S + \nabla_v J\circ du\circ J_S + \dot J\circ du\circ J_S + J\circ du\circ\dot J_S) =\\
&\underline{\nabla^2_{\tilde w,\cdot}v}_{[16']} + \underline{J\circ\nabla^2_{\tilde w,\cdot}v\circ J_S}_{[17']} + \underline{\nabla_w J\circ\nabla v\circ J_S}_{[5]} + \underline{\nabla^2_{w,v}J\circ du\circ J_S}_{[4']} +\\
&\underline{\nabla J(\nabla_{\tilde w}v; du\circ J_S)}_{[18]} + \underline{\nabla_v J\circ\nabla_{\tilde w}du\circ J_S}_{[3']} + \underline{\nabla_w\dot J\circ du\circ J_S}_{[7]} +\\
&\underline{\dot J\circ\nabla_{\tilde w}du\circ J_S}_{[6']} + \underline{\nabla_w J\circ du\circ\dot J_S}_{[9]} + \underline{J\circ\nabla_{\tilde w}du\circ\dot J_S}_{[8']} + \underline{J\circ du\circ\nabla_{\tilde w}\dot J_S}_{[13]}.
\end{aligned}
$$

Comparing the terms [16] and [16′], it follows that $\nabla^2_{\cdot,\tilde w}v - \nabla^2_{\tilde w,\cdot}v = R^X(du,w;v)$. A similar relation holds for the terms [17] and [17′]. The equality (4.3.2) of the lemma is obtained by subtracting (4.3.3) from (4.3.4). □

Lemma 4.3.2. *i) In the situation of Lemma 4.3.1, let $z^*\in S$ be a cusp-point. Consider $\tilde w$ as a section of TS with poles. Set $k := \mathsf{ord}_{z^*}du = \mathsf{dim}_{\mathbb C}(\mathscr N_u^{\mathsf{sing}})_{z^*}$ and choose a local complex coordinate z on S centered in z^*. Fix additionally $([v],\dot J_S,\dot J)\in T_{[u,J]}\mathscr M$ and $\psi\in\mathsf H^0_D(S,N_u^*\otimes K_S)$. Then locally in a neighborhood of z^**

$$
\begin{array}{rclllll}
z^k\cdot\tilde w(z) &=& w_0 & +z\cdot w_1 & +\cdots+z^k\cdot w_k & +z^k\cdot w^*(z), \\
v(z) &=& v_0 & +z\cdot v_1 & +\cdots+z^k\cdot v_k & +z^k\cdot v^*(z), \\
\psi(z) &=& \psi_0 & +z\cdot\psi_1 & +\cdots+z^k\cdot\psi_k & +z^k\cdot\psi^*(z),
\end{array}
\qquad (4.3.5)
$$

where $w^*(z)$, $v^*(z)$, and $\psi^*(z)$ are $L^{1,p}$-smooth local sections of the corresponding bundles vanishing at $z=0$.

ii) The polynomials in (4.3.5) can be considered as the order k jets of the following local holomorphic objects: a section of TS for $w_0+\cdots+z^k\cdot w_k$, a $(E_u)_{z^*}$-valued function for $v_0+\cdots+z^k\cdot v_k$, and resp. $(N_u^*)_{z^*}$-valued $(0,1)$-form for $\psi_0+\cdots+z^k\cdot\psi_k$. In particular, the coefficients can be considered as well-defined elements

$$
\begin{array}{rclcl}
w_i &=& \left(\frac{\partial}{\partial z}\right)^i (z^k\tilde w(z))_{|z=0} &\in& (T^*_{z^*}S)^{\otimes i-k}\otimes T_{z^*}S,\\
v_i &=& \nabla^i(v(z))_{|z=0} &\in& (T^*_{z^*}S)^{\otimes i}\otimes(E_u)_{z^*},\\
\psi_i &=& \nabla^i(\psi(z))_{|z=0} &\in& (T^*_{z^*}S)^{\otimes i}\otimes(N_u^*\otimes K_S)_{z^*}.
\end{array}
\qquad (4.3.6)
$$

Proof. *i)* It follows from *Definition 1.5.1* that du, considered as a *holomorphic* section of the bundle $\mathsf{Hom}_{\mathbb{C}}(TS, E_u)$, locally has the form $du(z) = z^k s(z)$ for some local holomorphic non-vanishing section s. Consequently, $\mathscr{O}([A]) = \mathscr{O}(k\cdot[z^*])$ in a neighborhood of z^*. Thus by *Lemma 4.3.1* $\tilde{w}$ can be locally represented in the form $\tilde{w}(z) = z^{-k}\cdot\hat{w}(z)$ for some local $L^{1,p}_{\mathsf{loc}}$-smooth section of TS. Equation (4.3.1) is equivalent to $\overline{\partial}\hat{w} = -\mathrm{i}\cdot z^k\cdot I_S$ and implies the estimate $|\overline{\partial}\hat{w}(z)| \leqslant |z^k|\cdot|I_S(z)|$. Now we use *Lemma 1.2.4*.

The same argument applies to $v(z)$ and $\psi(z)$. Indeed, equation (2.2.7) on v and relation (1.4.3) imply the inequality $|\overline{\partial}v(z)| \leqslant c\cdot|z^k|$ with some constant c. A similar inequality for $\psi(z)$ follows from (1.5.3).

ii) This part of the lemma can be reformulated in terms of the transformation of coefficients w_i, v_i, and ψ_i under the change of a local holomorphic coordinate z on S and local coordinates on X. The claim concerning the change of z is obvious.

Considering changes of coordinates on X we make the following observation. If $\boldsymbol{x}' = (x'_1, \dots, x'_{2n})$ and $\boldsymbol{x}'' = (x''_1, \dots, x''_{2n})$ are two systems of coordinates on X centered in $u(z^*)$, then $\boldsymbol{x}'' = L(\boldsymbol{x}') + Q(\boldsymbol{x}') + \cdots$, where $L : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ (resp. $Q : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$) is an appropriate linear (resp. quadratic) map, and so on. In particular, $\boldsymbol{x}'' - L(\boldsymbol{x}') = O(|\boldsymbol{x}'|^2)$. Consequently, for local frames $\partial_{\boldsymbol{x}'} = (\partial_{x'_1}, \dots, \partial_{x'_{2n}})$ and $\partial_{\boldsymbol{x}''} = (\partial_{x''_1}, \dots, \partial_{x''_{2n}})$ of TX we obtain the relation $\partial_{\boldsymbol{x}''}(\boldsymbol{x}') - L^{\mathsf{t}}(\partial_{\boldsymbol{x}''})(\boldsymbol{x}') = O(|\boldsymbol{x}'|)$. Thus, for the pulled-back frames $u^*\partial_{\boldsymbol{x}'}$ and $u^*\partial_{\boldsymbol{x}''}$ of E_u we have $u^*\partial_{\boldsymbol{x}'}(z) - L^{\mathsf{t}}(u^*\partial_{\boldsymbol{x}''})(z) = O(|z|^{k+1})$. This implies that the change of local coordinates on X induces only a linear transformation of the k-jet of v, *i.e.* the k-jet of v behaves like a k-jet of a $T_{u(z^*)}$-valued function. The same argumentations can be applied to the k-jet of ψ. □

Lemma 4.3.3. *For* $[u,J] \in \mathscr{M}$, $([w], I) \in \mathsf{H}^0_D(S, \mathscr{N}^{\mathsf{sing}}_n)$, $([v], \dot{J}_S, \dot{J}) \in T_{[u,J]}\mathscr{M}$, *and* $\psi \in \mathsf{H}^0_D(S, N_u \otimes K_S) \cong \mathsf{H}^1_D(S, N_u)^*$ *it follows that*

$$\Big\langle \psi, \Phi_{u,J}\big(([v], \dot{J}_S, \dot{J}),\ ([w], I_S)\big)\Big\rangle = \mathsf{Re}\,\mathsf{Res}_S(\psi\circ\nabla_{\tilde{w}}v), \tag{4.3.7}$$

where $\mathsf{Res}_S(\psi\circ\nabla_{\tilde{w}}v)$ *denotes the residual type sum*

$$\mathsf{Res}_S(\psi\circ\nabla_{\tilde{w}}v) := \sum_{du(z^*_i)=0} \lim_{\varepsilon\longrightarrow 0} \int_{|z-z^*_i|=\varepsilon} \psi\circ\nabla_{\tilde{w}}v \tag{4.3.8}$$

over all cusp-points $z^*_i \in S$ *of* u.

Moreover, if $\dot{J} = 0$ *and* $([v], \dot{J}_S) \in \mathsf{H}^0(S, \mathscr{N}^{\mathsf{sing}}_u)$, *then* $v = du(\tilde{v})$ *with* $\tilde{v} \in L^{1,p}(S, TS \otimes \mathscr{O}([A]))$ *and*

$$\Big\langle \psi, \Phi_{u,J}\big(([v], \dot{J}_S, 0),\ ([w], I_S)\big)\Big\rangle = \mathsf{Re}\,\mathsf{Res}_S(\psi\circ\nabla du(\tilde{w}, \tilde{v})). \tag{4.3.9}$$

Proof. First, we note that by *Lemma 4.3.2* the formulas (4.3.7–4.3.9) are well-defined.

Now, compute the subtraction of (4.3.2) from (4.2.11). The terms [5], [7], and [9] cancel. To simplify further terms we use the Bianchi identity and antisymmetry of $R^X(\cdot,\cdot;\cdot)$ in the first two arguments. The difference of terms $[1]+[2]+[4]-[1']-[2']-[4']$ is zero:

$$\underline{R^X(v, du; w)}_{[1]} - \underline{R^X(w, du; v)}_{[1']} + \underline{J\circ R^X(v, du\circ J_S; w)}_{[2]} - \tag{4.3.10}$$

$$\underline{J\circ R^X(w, du\circ J_S; v)}_{[2']} + \underline{\nabla^2_{v,w}J\circ du\circ J_S}_{[4]} - \underline{\nabla^2_{w,v}J\circ du\circ J_S}_{[4']} = \tag{4.3.11}$$

$$= R^X(v, du; w) + R^X(du, w; v) + J\circ R^X(v, du\circ J_S; w) + \tag{4.3.12}$$

$$J\circ R^X(du\circ J_S, w; v) + R^X(v, w; J\circ du\circ J_S) - J\circ R^X(v, w; du\circ J_S) = \tag{4.3.13}$$

$$= R^X(v, du; w) + R^X(du, w; v) + R^X(w, v; du) + \tag{4.3.14}$$

$$J \circ R^X(v, du \circ J_S; w) + J \circ R^X(du \circ J_S, w; v) + J \circ R^X(w, v; du \circ J_S) = 0. \tag{4.3.15}$$

In the differences $[3] - [3']$, $[6] - [6']$, and $[8] - [8']$ respectively, we use the relation

$$\nabla(w) = \nabla(du(\tilde{w})) = \nabla_{\tilde{w}} du + du \circ \nabla\tilde{w}. \tag{4.3.16}$$

This yields

$$\underline{\nabla_v J \circ \nabla w \circ J_S}_{[3]} - \underline{\nabla_v J \circ \nabla_{\tilde{w}} du \circ J_S}_{[3']} = \underline{\nabla_v J \circ du \circ \nabla\tilde{w} \circ J_S}_{[3'']}, \tag{4.3.17}$$

and similar equalities for $[6] - [6']$ and $[8] - [8']$. Thus we obtain

$$\nabla_{([v], \dot{J}_S, \dot{J})}(\widehat{D})([w], I_S) = \underline{\nabla_v J \circ du \circ \nabla\tilde{w} \circ J_S}_{[3'']} + \underline{\dot{J} \circ du \circ \nabla\tilde{w} \circ J_S}_{[6'']} + \tag{4.3.18}$$

$$\underline{J \circ du \circ \nabla\tilde{w} \circ \dot{J}_S}_{[8'']} + \underline{\nabla_v J \circ du \circ I_S}_{[10]} + \underline{J \circ \nabla v \circ I_S}_{[11]} + \underline{\dot{J} \circ du \circ I_S}_{[12]} \tag{4.3.19}$$

$$- \underline{J \circ du \circ \nabla_{\tilde{w}} \dot{J}_S}_{[13]} + \underline{\nabla v \circ \nabla\tilde{w}}_{[14]} + \underline{J \circ \nabla v \circ \nabla\tilde{w} \circ J_S}_{[15]} - \underline{D_{u,J}(\nabla_{\tilde{w}} v)}_{[D]}. \tag{4.3.20}$$

Further simplification uses the relation $I_S = J_S \circ \nabla\tilde{w} - \nabla\tilde{w} \circ J_S$. This gives

$$\underline{\nabla_v J \circ du \circ \nabla\tilde{w} \circ J_S}_{[3'']} + \underline{\nabla_v J \circ du \circ I_S}_{[10]} = \nabla_v J \circ du \circ \nabla\tilde{w} \circ J_S + \tag{4.3.21}$$

$$\nabla_v J \circ du \circ (J_S \circ \nabla\tilde{w} - \nabla\tilde{w} \circ J_S) = \underline{\nabla_v J \circ du \circ J_S \circ \nabla\tilde{w}}_{[3''']}, \tag{4.3.22}$$

and similarly,

$$\underline{\dot{J} \circ du \circ \nabla\tilde{w} \circ J_S}_{[6'']} + \underline{\dot{J} \circ du \circ I_S}_{[12]} = \underline{\dot{J} \circ du \circ J_S \circ \nabla\tilde{w}}_{[6''']}, \tag{4.3.23}$$

$$\underline{J \circ \nabla v \circ I_S}_{[11]} + \underline{J \circ \nabla v \circ \nabla\tilde{w} \circ J_S}_{[15]} = \underline{J \circ \nabla v \circ J_S \circ \nabla\tilde{w}}_{[15']}. \tag{4.3.24}$$

Now we put together the terms $[3''']$, $[6''']$, $[14]$, and $[15']$. Because of the relation

$$\nabla v + J \circ \nabla v \circ J_S + \nabla_v J \circ du \circ J_S + \dot{J} \circ du \circ J_S + J \circ du \circ \dot{J}_S = 0 \tag{4.3.25}$$

this yields

$$\underline{\nabla_v J \circ du \circ J_S \circ \nabla\tilde{w}}_{[3''']} + \underline{\dot{J} \circ du \circ J_S \circ \nabla\tilde{w}}_{[6''']} + \underline{\nabla v \circ \nabla\tilde{w}}_{[14]} + \underline{J \circ \nabla v \circ J_S \circ \nabla\tilde{w}}_{[15']} = \tag{4.3.26}$$

$$\big(\nabla_v J \circ du \circ J_S + \dot{J} \circ du \circ J_S + \nabla v + J \circ \nabla v \circ J_S\big) \circ \nabla\tilde{w} = -\underline{J \circ du \circ \dot{J}_S \circ \nabla\tilde{w}}_{[8''']}. \tag{4.3.27}$$

Finally, we conclude that outside the zero-set of du one has

$$\nabla_{([v], \dot{J}_S, \dot{J})}(\widehat{D})([w], I_S) = \tag{4.3.28}$$

$$J \circ du \circ \big(\underline{\nabla\tilde{w} \circ \dot{J}_S}_{[8'']} - \underline{\dot{J}_S \circ \nabla\tilde{w}}_{[8''']} - \underline{\nabla_{\tilde{w}} \dot{J}_S}_{[13]}\big) - \underline{D_{u,J}(\nabla_{\tilde{w}} v)}_{[D]}. \tag{4.3.29}$$

Note that $\psi \circ J \circ du = \psi \circ du \circ J_S = 0$, since ψ vanishes on $du(TS)$. Consequently,

$$\Big\langle \psi, \Phi_{u,J}\big(([v], \dot{J}_S, \dot{J}),\ ([w], I_S)\big) \Big\rangle = \mathsf{Re} \int_S \psi \circ \Big(-\nabla_{([v], \dot{J}_S, \dot{J})} \widehat{D}\Big)([w], I_S) = \tag{4.3.30}$$

$$\mathsf{Re} \lim_{\varepsilon \to 0} \int_{S \setminus \cup \Delta(z_i, \varepsilon)} \psi \circ \Big(-\nabla_{([v], \dot{J}_S, \dot{J})} \widehat{D}\Big)([w], I_S) = \mathsf{Re} \lim_{\varepsilon \to 0} \int_{S \setminus \cup \Delta(z_i, \varepsilon)} \psi \circ D_{u,J}(\nabla_{\tilde{w}} v). \tag{4.3.31}$$

Integrating by parts and using $D\psi = 0$ we obtain the desired formula (4.3.7).

To obtain (4.3.9) we use (4.3.16) and relation $\psi \circ du = 0$. $\square$

Now we can describe the structure of $\Phi_{u,J}$ for cusp-curves. Here we restrict ourselves to the case when (X, J) is an almost complex surface. The point is that, unlike to the higher dimensional situation, in this dimension there are *topological* reasons for the existence of cusp-curves, see *Lemma 2.3.4*.

Other than the restriction on dimension, our setting is as follows. $[u, J] \in \mathscr{M}$ is a J-holomorphic curve with $\mathsf{H}^1(S, N_u) \cong \mathbb{R}$, $z^* \in S$ a cuspidal point, $k := \mathsf{ord}_{z^*} du$, z a local complex coordinate centered at z^*, J^* a local (integrable) complex structure in a neighborhood of $u(z^*)$, and (w^1, w^2) a local system of J^*-holomorphic coordinates on X centered at $u(z^*)$. Finally, we fix some *non-zero* $\psi \in \mathsf{H}^0(S, N_u^* \otimes K_S) \cong \mathsf{H}^1(S, N_u)^*$ and denote by $\mathsf{ord}_{z^*}\psi$ the order of vanishing of ψ at z^*.

Lemma 4.3.4. *i) After a polynomial transformation of the coordinates (w^1, w^2), chosen above, the map u will have the form*

$$u(z) = \big(z^{k+1} P_1(z), z^{k+l+2} P_2(z)\big) + z^{2k+1} g(z), \tag{4.3.32}$$

such that $0 \leqslant l \leqslant k$, P_1 is a polynomial of degree $\leqslant k$ with $P_1(0) \neq 0$, P_2 is a polynomial of degree $\leqslant k-l-1$, trivial if $l = k$ or with $P_2(0) \neq 0$ otherwise, and $g(z)$ is an $L^{1,p}$-smooth $\mathbb{C}^2$-valued function.

ii) The integers $\mathsf{ord}_{z^*}\psi$ and l do not depend on the particular choice of coordinates (w^1, w^2) and $\psi \in \mathsf{H}^0(S, N_u^* \otimes K_S)$. For the restriction of $\Phi_{u,J}$ on the stalk $(\mathscr{N}_u^{\mathsf{sing}})_{z^*} \subset \mathsf{H}^0(S, \mathscr{N}_u^{\mathsf{sing}})$, it follows that

$$\begin{array}{rcl} \mathsf{ind}_+\big(\Phi_{u,J}|_{(\mathscr{N}_u^{\mathsf{sing}})_{z^*}}\big) & = & \mathsf{ind}_-\big(\Phi_{u,J}|_{(\mathscr{N}_u^{\mathsf{sing}})_{z^*}}\big) = \\ \mathsf{S\text{-}ind}\big(\Phi_{u,J}|_{(\mathscr{N}_u^{\mathsf{sing}})_{z^*}}\big) & = & \mathsf{max}(0, k - l - \mathsf{ord}_{z^*}\psi). \end{array} \tag{4.3.33}$$

iii) If z_1^* and z_2^* are distinct cusp-points of $u : S \to X$, then the stalks $(\mathscr{N}_u^{\mathsf{sing}})_{z_1^*}$ and $(\mathscr{N}_u^{\mathsf{sing}})_{z_2^*}$ are Φ-orthogonal, i.e.

$$\Phi_{u,J}\big((\mathscr{N}_u^{\mathsf{sing}})_{z_1^*}, (\mathscr{N}_u^{\mathsf{sing}})_{z_2^*}\big) = 0. \tag{4.3.34}$$

Proof. *Part i)* follows immediately from *Lemma 1.2.5*. It simply says that if $\mathsf{ord}_{z^*} du = k$, then the jet $j^{2k+1}u$ is well-defined and *holomorphic*, *i.e.* can be represented by a complex polynomial. Note that the theorem of [Mi-Wh] (see *Lemma 1.2.1*) says that *topologically* one can also define higher terms which determine the whole behavior of u at z^*.

Part iii) can be easily obtained from (4.3.8) and (4.3.9). It remains to consider

Part ii). First, we observe that the integer l is the secondary cusp index of u at z^* (see *Definition 3.3.1*). It follows then from the results of *Paragraph 3.3* that this integer is well defined and independent of the choice of (w^1, w^2). The independence of $\mathsf{ord}_{z^*}\psi$ of the choice of (w^1, w^2) and ψ is obvious.

Let J^* and (w^1, w^2) be a complex structure and J^*-holomorphic coordinates in a neighborhood of $u(z^*)$, such that $J^*(u(z^*)) = J(u(z^*))$ and u has the local form (4.3.32). Differentiating (4.3.32) we see that in the coordinates (w^1, w^2)

$$du(z) = \big(z^k P_1'(z), z^{k+l+1} P_2'(z)\big) + z^{2k} g'(z), \tag{4.3.35}$$

with polynomials

$$P_1'(z) = (k+1)P_1(z) + z\tfrac{d}{dz}P_1(z) \quad \text{and} \quad P_2'(z) = (k+l+2)P_2(z) + z\tfrac{d}{dz}P_2(z), \tag{4.3.36}$$

of degree $\leqslant k$ and $\leqslant k-l-1$ respectively and with $g'(z) = (2k+1)g(z)dz + z dg(z)$ being L^p-bounded.

From the definition of the Nijenhuis torsion tensor N_J of J we obtain a pointwise estimate $|\bar\partial_J w_\alpha| \leqslant |N_J|$. Further,

$$\bar\partial(w_\alpha \circ u) = (dw_\alpha \circ du)^{(0,1)} = \bar\partial_J w_\alpha \circ du, \tag{4.3.37}$$

since u is J-holomorphic. Consequently, we obtain a pointwise estimate

$$|\bar\partial(w_\alpha \circ u)(z)| \leqslant c \cdot |z^k| \tag{4.3.38}$$

with some constant c. Let $\{e^*_\alpha\}_{\alpha=1,2}$ be the local J^*-complex frame of T^*X dual to the frame $\{dw_\alpha\}_{\alpha=1,2}$. Then there exists a local J-complex frame $\{e_\alpha\}_{\alpha=1,2}$ of T^*X with pointwise estimates

$$|e^*_\alpha(w) - e_\alpha(w)| \leqslant c \cdot |w| \quad \text{and} \quad |\nabla e^*_\alpha(w) - \nabla e_\alpha(w)| \leqslant c, \tag{4.3.39}$$

where $|w|^2 = |w_1|^2 + |w_2|^2$ and c is some constant. Using (4.3.35–4.3.39) and the estimates $|u(z)| \leqslant c \cdot |z^{k+1}|$ and $|du(z)| \leqslant c \cdot |z^k|$ we conclude that

a) $\boldsymbol{e}_\alpha := u^* e_\alpha$ is a local complex frame of $E_u = u^*TX$ with a pointwise estimate

$$|\bar\partial_{u,J} \boldsymbol{e}_\alpha(z)| \leqslant c \cdot |z^k|; \tag{4.3.40}$$

b) du, considered as a section of $E_u \otimes T^*S$ with the frame $\boldsymbol{e}_\alpha \otimes dz$, has local form (4.3.35), possibly with another $g'(z) \in L^p$. Moreover, since du is a holomorphic section and $\boldsymbol{e}_\alpha$ are sufficiently regular, this new $g'(z)$ is C^1-smooth. Further, since $zdg(z) = g'(z) - (2k+1)zg(z)dz$ is continuous and $dg(z) \in L^p$ with $p > 2$, we conclude that $zdg(z)$ vanishes at $z = 0$. This gives an additional relation $g'(0) = 0$.

Differentiating (4.3.35) we obtain that in the frame $\boldsymbol{e}_\alpha \otimes dz^2$

$$\nabla du(z) = \big(z^{k-1} P''_1(z), z^{k+l} P''_2(z)\big) + z^{2k-1} g''(z), \tag{4.3.41}$$

with polynomials

$$P''_1(z) = k\,P'_1(z) + z\tfrac{d}{dz} P'_1(z) \quad \text{and} \quad P''_2(z) = (k+l+1)P_2(z) + z\tfrac{d}{dz} P'_2(z), \tag{4.3.42}$$

of degree $\leqslant k$ and $\leqslant k-l-1$ respectively and with

$$g''(z) = (2k+1)g'(z) \otimes dz + z\nabla g'(z) \tag{4.3.43}$$

continuous and vanishing at $z = 0$. By our construction, $P''_1(0) = (k+1)kP_1(0) \neq 0$ and $P''_2(0) = (k+l+2)(k+l+1)P_1(0)$ vanishes if and only if $l = k$.

Since the projection $\mathsf{pr} : E_u \to N_u$ is obtained as the quotient with respect to the image of $z^{-k}du \sim (P'_1(z), z^{l+1}P'_2(z))$, we have the following form for the composition:

$$\mathsf{pr} \circ \nabla du(z) = P'''(z) + g'''(z), \tag{4.3.44}$$

where $P'''(z)$ is a polynomial $P'''(z)$ of degree $\leqslant k-l-1$ given by the relation

$$z^{k+l}P'''(z) = z^{k+l}P''_2(z) - \frac{z^{k+l+1}P'_2(z) \cdot z^{k-1}P''_1(z)}{z^k P'_1(z)} + o(z^{2k-1}). \tag{4.3.45}$$

In particular, $P'''(0) = (k+l+1)(l+1)P_2(0)$ vanishes if and only if $l = k$.

Denoting $\nu := \mathsf{ord}_{z^*}\psi$ we obtain that

$$\psi \circ \nabla du(z) = az^{k+l+\nu} + o(z^{k+l+\nu}) \tag{4.3.46}$$

with a vanishing if and only if $l = k$. The proof of part *ii)* of the lemma can be now finished using the following algebraic result. $\square$

Lemma 4.3.5. *For a given polynomial $P(z) = a_0 + a_1 z + \cdots + a_{k-l-1} z^{k-l-1}$ with $a_0 \neq 0$ and $0 \leqslant l < k$ the quadratic form*

$$(w_0, \ldots, w_k) \in \mathbb{C}^{k+1} \mapsto \mathsf{Re}\,\mathsf{Res}_{z=0}\left(\frac{z^{k+l}\, P(z) \left(\sum_{i=0}^{k} w_i\, z^i\right)^2}{z^{2k}} dz\right) \in \mathbb{R} \tag{4.3.47}$$

is equivalent to the quadratic form

$$(w_0, \ldots, w_k) \in \mathbb{C}^{k+1} \mapsto \mathsf{Re}\,\mathsf{Res}_{z=0}\left(\frac{z^{k+l}\, a_0 \left(\sum_{i=0}^{k} w_i\, z^i\right)^2}{z^{2k}} dz\right) \in \mathbb{R} \tag{4.3.48}$$

and satisfies the index relations

$$\mathsf{ind}_+ Q = \mathsf{ind}_- Q = \mathsf{S}\text{-}\mathsf{ind}\, Q = k - l. \tag{4.3.49}$$

4.4. **Critical points and cusp-curves in the moduli space.** Recall that in *Lemma 4.3.4* we found two obstructions for existence of saddle points. They are encoded in the secondary cusp-indices l_i of cusp-points z_i^* of u (see *Definition 3.3.1*) and the vanishing order at z_i^* of a generic $\psi \in \mathsf{H}^0_D(S, N_u \otimes K_S) \cong \left(\mathsf{H}^1_D(S, N_u)\right)^*$. The behavior l_i under deformation was studied in *Paragraph 3.3*. In this paragraph we describe the behavior of $\mathsf{H}^0_D(S, N_u \otimes K_S)$. Our main interest is, of course, $[u, J] \in \mathscr{M}$ with $\mathsf{dim}_{\mathbb{R}} \mathsf{H}^1_D(S, N_u) = 1$, because these are candidates for saddle points. We start with

Lemma 4.4.1. *Let $\boldsymbol{k} = (k_1, \ldots, k_m)$ and $h^1 \in \mathbb{N}$ be given. Then the set*

$$\widehat{\mathscr{M}}_{=\boldsymbol{k},h^1} := \left\{(u, J_S, J; \boldsymbol{z}) \in \widehat{\mathscr{M}}_{=\boldsymbol{k}} \,:\, \mathsf{dim}_{\mathbb{R}} \mathsf{H}^1_D(S, N_u) = h^1 \right\} \subset \widehat{\mathscr{M}}_{=\boldsymbol{k}} \tag{4.4.1}$$

is a $C^{\ell-1}$-smooth submanifold of codimension $h^0 \cdot h^1$ where $h^0 := \mathsf{dim}_{\mathbb{R}} \mathsf{H}^0_D(S, N_u)$.

The set $\widehat{\mathscr{M}}_{=\boldsymbol{k},h^1}$ is $\mathbf{G}$-invariant and the projection $\mathsf{pr} : \widehat{\mathscr{M}}_{=\boldsymbol{k},h^1} \longrightarrow \mathscr{M}_{=\boldsymbol{k},h^1} := \widehat{\mathscr{M}}_{=\boldsymbol{k},h^1}/\mathbf{G}$ is a $C^{\ell-1}$-smooth principle $\mathbf{G}$-bundle.

Remark. The definition (1.5.1) of $\mathscr{N}_u$ and the index formula (2.2.6) imply that $h^0 = h^1 + 2(\mu + (g-1)(3-n) - |\boldsymbol{k}|)$, where $n = \frac{1}{2}\mathsf{dim}_{\mathbb{R}} X$ and $\mu := \langle c_1(X), [u(S)]\rangle$. So $h^0 = \mathsf{dim}_{\mathbb{R}} \mathsf{H}^0_D(S, N_u)$ is constant along $\widehat{\mathscr{M}}_{=\boldsymbol{k},h^1}$ and

$$\widehat{\mathscr{M}}_{=\boldsymbol{k}} = \bigsqcup_{h^1=0}^{\infty} \widehat{\mathscr{M}}_{=\boldsymbol{k},h^1} \tag{4.4.2}$$

is a stratification of $\widehat{\mathscr{M}}_{=\boldsymbol{k}}$ indexed by $h^1 = \mathsf{dim}_{\mathbb{R}} \mathsf{H}^1_D(S, N_u)$. Taking the $\mathbf{G}$-quotients, we obtain a similar stratification of $\mathscr{M}_{=\boldsymbol{k}}$. Another stratification, more interesting for our purpose, is

$$\mathscr{M}_{h^1} = \bigsqcup_{\boldsymbol{k}} \mathscr{M}_{=\boldsymbol{k},h^1} \tag{4.4.3}$$

with $h^1 = 1$. Note that if for given $\boldsymbol{k}$ and h^1 the expected value of h^0 is negative, then $\widehat{\mathscr{M}}_{=\boldsymbol{k},h^1}$ is empty.

Proof. Consider the Banach bundles $L^{1,p}(S, N)$, $L^p_{(0,1)}(S, N)$ over $\widehat{\mathscr{M}}_{=\boldsymbol{k}}$, and the bundle homomorphism $D^N : L^{1,p}(S, N) \to L^p_{(0,1)}(S, N)$ constructed in *Lemma 3.2.7*. Then $\mathsf{H}^i_D(S, N_u)$ is the (co)kernel of D^N. From *Lemma 1.3.1* we obtain the map

$$\nabla_{(v, \dot{J}_S, \dot{J})} D^N : \mathsf{H}^0_D(S, N_u) \to \mathsf{H}^1_D(S, N_u), \tag{4.4.4}$$

which is bilinear in $(v, \dot{J}_S, \dot{J}) \in T_{(u,J_S,J)}\widehat{\mathscr{M}}_{=\boldsymbol{k}}$ and $w \in \mathsf{H}^0_D(S, N_u)$. It is not difficult to see that the map (4.4.4) can be computed using (4.2.11) and that it coincides with the restriction of Φ from (4.1.1) to the corresponding spaces.

The key point of the proof is to show the surjectivity of the induced map

$$\Phi : T_{(u,J_S,J)}\widehat{\mathscr{M}_{=\boldsymbol{k}}} \longrightarrow \mathsf{Hom}_{\mathbb{R}}\big(\mathsf{H}^0_D(S,N_u),\, \mathsf{H}^1_D(S,N_u)\big) \tag{4.4.5}$$

Then the claim of the lemma will follow from the implicit function theorem.

Fix bases $(w_1,\dots,w_{h^0})$ of $\mathsf{H}^0_D(S,N_u)$ and $(\psi_1,\dots,\psi_{h^1})$ of $\mathsf{H}^0_D(S,N_u\otimes K_S)\cong\big(\mathsf{H}^1(S,N_u)\big)^*$. The last isomorphism is the Serre duality from *Lemma 1.5.1*. We must find tangent vectors $(v_{ij},\dot J_{S,ij},\dot J_{ij})\in T_{(u,J_S,J)}\widehat{\mathscr{M}_{=\boldsymbol{k}}}$, $i=1,\dots,h^0$, $j=1,\dots,h^1$ obeying the relation

$$\big\langle \psi_{j'},\,\Phi\big((v_{ij},\dot J_{S,ij},\dot J_{ij}),w_{i'}\big)\big\rangle=\delta_{ii'}\delta_{jj'} \tag{4.4.6}$$

with $\langle\cdot,\cdot\rangle$ denoting the pairing from (1.5.7).

The main idea is to find solutions of (4.4.6) in the special form such that v_{ij} and $\dot J_{S,ij}$ are identically zero, and $\dot J_{ij}$ vanish along $u(S)$ and in a neighborhood of all cusp-points on $u(S)$. This assumption implies that all the terms in (4.2.11) except [7] vanish. Thus (4.4.6) reduces to

$$\mathsf{Re}\int_S \psi_{j'}\circ\nabla_{w_{i'}}\dot J_{ij}\circ du\circ J_S=\delta_{ii'}\delta_{jj'}. \tag{4.4.7}$$

From this point we can use the arguments either of *Lemma 2.1.2* or *Lemma 3.2.4*. Note that we can arrange $\dot J_{ij}$ to have support in any given open subset $U\subset X$ with $U\cap u(S)\neq\varnothing$. □

The big freedom in the choice of $\dot J_{ij}$ implies the following

Corollary 4.4.2. *Let $\mathsf{dim}_{\mathbb{R}}X=4$, i.e. X is an almost complex surface. Then the intersection of $\widehat{\mathscr{M}_{=\boldsymbol{k},\boldsymbol{l}}}$ and $\widehat{\mathscr{M}_{=\boldsymbol{k},h^1}}$ is transversal, so that the set*

$$\widehat{\mathscr{M}_{=\boldsymbol{k},\boldsymbol{l},h^1}}:=\widehat{\mathscr{M}_{=\boldsymbol{k},\boldsymbol{l}}}\cap\widehat{\mathscr{M}_{=\boldsymbol{k},h^1}} \tag{4.4.8}$$

is a $C^{\ell-1}$-smooth submanifold of $\widehat{\mathscr{M}_{=\boldsymbol{k}}}$ of codimension $2|\boldsymbol{l}|+h^0\cdot h^1$. A similar result also holds for $\mathscr{M}_{=\boldsymbol{k},\boldsymbol{l},h^1}:=\widehat{\mathscr{M}_{=\boldsymbol{k},\boldsymbol{l},h^1}}/\mathbf{G}=\mathscr{M}_{=\boldsymbol{k},\boldsymbol{l}}\cap\mathscr{M}_{=\boldsymbol{k},h^1}$.

Now we will study the behavior of zeros of a non-trivial $\psi\in\mathsf{H}^0_D(S,N_u\otimes K_S)$ for $[u,J]\in\mathscr{M}_{=\boldsymbol{k},h^1=1}$. Note that, modifying the construction from *Lemma 4.4.3*, we obtain a bundle $N^*\otimes K_S$ over $\widehat{\mathscr{M}_{=\boldsymbol{k}}}\times S$, $C^{\ell-1}$-smooth Banach bundles $L^{1,p}(S,N^*\otimes K_S)$ and $L^p_{(0,1)}(S,N^*\otimes K_S)$ over $\widehat{\mathscr{M}_{=\boldsymbol{k}}}$, and a $C^{\ell-1}$-smooth bundle homomorphism

$$(D^N)^*: L^{1,p}(S,N^*\otimes K_S)\to L^p_{(0,1)}(S,N^*\otimes K_S). \tag{4.4.9}$$

Since the kernel of $(D^N)^*$ is of constant dimension on each $\widehat{\mathscr{M}_{=\boldsymbol{k},h^1}}$, we obtain a $C^{\ell-1}$-smooth bundle $\mathsf{H}^0_D(S,N^*\otimes K_S)$ of $\mathsf{rank}_{\mathbb{R}}=h^1$ on $\widehat{\mathscr{M}_{=\boldsymbol{k},h^1}}$. This means that there exists a (local) frame $\psi_1,\dots,\psi_{h^1}$ of $\mathsf{H}^0_D(S,N^*\otimes K_S)$ which depends $C^{\ell-1}$-smoothly on $(u,J_S,J)\in\widehat{\mathscr{M}_{=\boldsymbol{k},h^1}}$.

In the particular case $h^1=1$ we obtain a (local) $C^{\ell-1}$-smooth family of non-trivial $\psi\in\mathsf{H}^0_D(S,N^*\otimes K_S)$ such that for every (u,J_S,J) the corresponding ψ is defined uniquely up to a constant factor. *Lemma 1.2.4* ensures that the zero-divisor of such ψ is well-defined and has degree $c_1(N^*\otimes K_S)$. By *Lemma 2.3.4*, the possible range for $c_1(N^*\otimes K_S)$ is the interval between 0 and $g-1$. We are interested in the distribution of the zeros of ψ, especially at cusp-points of u. For a given $\boldsymbol{k}=(k_1,\dots,k_m)$ we consider m-tuples $\boldsymbol{\nu}=(\nu_1,\dots,\nu_m)$ with $0\leqslant\nu_i\leqslant k_i$, $i=1,\dots,m$. Denote $|\boldsymbol{\nu}|:=\sum_{i=1}^m\nu_i$.

Lemma 4.4.3. *i) The set* $\widehat{\mathscr{M}}_{=\boldsymbol{k},\boldsymbol{\nu}} := \big\{(u,J_S,J)\in\widehat{\mathscr{M}}_{=\boldsymbol{k},h^1=1}\ :\ \mathsf{ord}_{z_i^*}\psi\geqslant\nu_i\big\}\subset\widehat{\mathscr{M}}_{=\boldsymbol{k},h^1=1}$ *is a* $C^{\ell-1}$*-smooth submanifold of codimension* $2(n-1)|\boldsymbol{\nu}|$, $n=\frac{1}{2}\mathsf{dim}_{\mathbb{R}}X$.

ii) Let $n=2$, *i.e.* X *is an almost complex surface. Then the intersection of* $\widehat{\mathscr{M}}_{=\boldsymbol{k},\boldsymbol{l}}$ *and* $\widehat{\mathscr{M}}_{=\boldsymbol{k},\boldsymbol{\nu}}$ *is transversal, so that the set*

$$\widehat{\mathscr{M}}_{=\boldsymbol{k},\boldsymbol{l},\boldsymbol{\nu}} := \widehat{\mathscr{M}}_{=\boldsymbol{k},\boldsymbol{l},h^1=1}\cap\widehat{\mathscr{M}}_{=\boldsymbol{k},\boldsymbol{\nu}}\ \subset\ \widehat{\mathscr{M}}_{=\boldsymbol{k},h^1=1} \tag{4.4.10}$$

is a $C^{\ell-1}$*-smooth submanifold of codimension* $2|\boldsymbol{l}|+|\boldsymbol{\nu}|$.

iii) Similar results hold for $\mathscr{M}_{=\boldsymbol{k},\boldsymbol{l},\boldsymbol{\nu}} := \widehat{\mathscr{M}}_{=\boldsymbol{k},\boldsymbol{l},\boldsymbol{\nu}}/\mathbf{G} = \mathscr{M}_{=\boldsymbol{k},\boldsymbol{l},h^1=1}\cap\mathscr{M}_{=\boldsymbol{k},\boldsymbol{\nu}}$.

Proof. *i)*. Fix $(u_0,J_{S,0},J_0)\in\widehat{\mathscr{M}}_{=\boldsymbol{k},\boldsymbol{\nu}}$. Let z_i be a local J_S-holomorphic coordinate on $\widehat{\mathscr{M}}_{=\boldsymbol{k},h^1=1}$ in the sense of *Definition 3.2.3*, centered at the cusp-point z_i^* of (u,J_S,J), $i=1,\ldots,m$. Further, let ψ be a local $C^{\ell-1}$-smooth family of non-trivial elements of $\mathsf{H}^0_D(S,N^*\otimes K_S)$. Then by *Lemma 4.3.2*, for each $i=1,\ldots,m$ and each $(u,J_S,J)\in\widehat{\mathscr{M}}_{=\boldsymbol{k},h^1=1}$ we can construct the jets $j^{k_i}\psi=\sum_{j=0}^{k_i}\psi_{i,j}\cdot z_i^j$ of ψ at z_i^*.

Repeating the arguments used in the proof of *Lemma 3.2.3* we can show that the coefficients $\psi_{i,j}\in(T^*_{z_i^*})^{\otimes j}\otimes(N^*\otimes K_S)_{z_i^*}$ depend $C^{\ell-1}$-smoothly on $(u,J_S,J)\in\widehat{\mathscr{M}}_{=\boldsymbol{k},h^1=1}$. This means that $\widehat{\mathscr{M}}_{=\boldsymbol{k},\boldsymbol{\nu}}$ is the zero set of the (locally defined) function $\Upsilon^\psi_{\boldsymbol{\nu}}$ on $\widehat{\mathscr{M}}_{=\boldsymbol{k},h^1=1}$ given by the first ν_i coefficients of each $j^{k_i}\psi$, $i=1,\ldots,m$, *i.e.*

$$\Upsilon^\psi_{\boldsymbol{\nu}}(u,J_S,J)=(\psi_{1,0},\ldots,\psi_{1,\nu_1-1},\ldots,\psi_{m,0},\ldots,\psi_{m,\nu_m-1}). \tag{4.4.11}$$

Consequently, it is sufficient to show the surjectivity of the differential $d\Upsilon^\psi_{\boldsymbol{\nu}}$ at the fixed $(u_0,J_{S,0},J_0)$. But first we must compute $d\Upsilon^\psi_{\boldsymbol{\nu}}$ for a given $(v,\dot J_S,\dot J)\in T_{(u_0,J_{S,0},J_0)}\widehat{\mathscr{M}}_{=\boldsymbol{k},h^1=1}$. Let $\gamma(t)=(u_t,J_{S,t},J_t)$ be a curve in $\widehat{\mathscr{M}}_{=\boldsymbol{k},h^1=1}$ which starts at $(u_0,J_{S,0},J_0)$ and has the tangent vector $(v,\dot J_S,\dot J)$ at $t=0$. Then we obtain a family ψ_t of non-trivial $\psi_t\in\mathsf{H}^0_D(S,N^*_{u_t}\otimes K_S)$. In particular, for each t we obtain the relation $D^*_t\psi_t=0$, where D^*_t denotes the operator $D^{N^*\otimes K_S}$ corresponding to $(u_t,J_{S,t},J_t)$.

Fix some symmetric connections on X and S. As in *Paragraph 4.2*, we obtain induced connections for all (usual and Banach) bundles involved in our computations. We use the same notation ∇ for all these connections, in particular, for the connection in the bundle $L^{1,p}(S,N_u)$ with the fiber $L^{1,p}(S,N_{u_t})$ over $(u_t,J_{S,t},J_t)$. Hence for any $w_0\in L^{1,p}(S,N_{u_0})$ we can construct a family $w_t\in L^{1,p}(S,N_{u_t})$ which is covariantly constant. This yields a covariantly constant trivialization of the Banach bundle $L^{1,p}(S,N_u)$ along γ.

For every such family w_t we have the relation

$$\langle w_t,D^*_t\psi_t\rangle=0. \tag{4.4.12}$$

Vice versa, a family $\psi_t\in L^{1,p}(S,N^*_{u_t}\otimes K_S)$ lies in $\mathsf{H}^1_D(S,N^*_{u_t}\otimes K_S)$ if (4.4.12) holds. Rewrite (4.4.12) in the form

$$\langle\psi_t,D_tw_t\rangle=0 \tag{4.4.13}$$

with D_t denoting the operator $D^N_{u_t,J_t}:L^{1,p}(S,N_{u_t})\to L^p_{(0,1)}(S,N_{u_t})$. After covariant differentiation in t we obtain $\langle\dot\psi_t,D_tw_t\rangle+\langle\psi_t,(\nabla_{(v_t,\dot J_{S,t},\dot J_t)}D_t)w_t\rangle=0$. The latter is equivalent to

$$\langle D^*_t\dot\psi_t,w_t\rangle+\langle\psi_t,(\nabla_{(v_t,\dot J_{S,t},\dot J_t)}D_t)w_t\rangle=0. \tag{4.4.14}$$

Now we can give the description of $d\Upsilon^\psi_{\boldsymbol{\nu}}$ at $(u_0,J_{S,0},J_0)\in\widehat{\mathscr{M}}_{=\boldsymbol{k},h^1=1}$. For a given tangent vector $(v,\dot J_S,\dot J)$ we find $\dot\psi\in L^{1,p}(S,N_{u_0}\otimes K_S)$ such that (4.4.14) holds for every

$w \in L^{1,p}(S, N_{u_0} \otimes K_S)$. The existence of such $\dot\psi$ is equivalent to the condition that $(v, \dot J_S, \dot J)$ is tangent to $\widehat{\mathscr{M}}_{=\boldsymbol{k}, h^1=1}$. Such $\dot\psi$ is unique up to addition of $\psi \in \mathsf{H}^0_D(S, N_{u_0} \otimes K_S)$. The jets $j^{k_i}\dot\psi = \sum_{j=0}^{k_i} \dot\psi_{i,j} \cdot z_i^j$ of such $\dot\psi$ at z_i^* are well-defined and

$$d\Upsilon^\psi_{\boldsymbol{\nu}}(v, \dot J_S, \dot J) = (\dot\psi_{1,0}, \ldots, \dot\psi_{1,\nu_1-1}, \ldots, \dot\psi_{m,0}, \ldots, \dot\psi_{m,\nu_m-1}). \tag{4.4.15}$$

$d\Upsilon^\psi_{\boldsymbol{\nu}}$ is independent of the choice of $\dot\psi$ provided $(u_0, J_{S,0}, J_0) \in \widehat{\mathscr{M}}_{=\boldsymbol{k},\boldsymbol{\nu}}$.

To show the surjectivity of $d\Upsilon^\psi_{\boldsymbol{\nu}}$ we must invert the construction above. Let $j^{\nu_i-1}\dot\psi$ be given jets. Extend them to jets $j^{k_i}\dot\psi$. Note that by definition the operator $D_0^* = D^{N^*\otimes K_S}_{u_0,J_0}$ has the form $\overline\partial^{N^*\otimes K_S}_{u_0,J_0} + R^{N^*\otimes K_S}_{u_0,J_0}$ where

$$R^{N^*\otimes K_S}_{u_0,J_0} : N^* \otimes K_S \to N^* \otimes K_S \otimes \Lambda^{(0,1)} \tag{4.4.16}$$

is a continuous bundle homomorphism. Consider the equations

$$z_i^{-k_i}\big(\overline\partial^{N^*\otimes K_S}_{u_0,J_0} + R^{N^*\otimes K_S}_{u_0,J_0}\big)\big(j^{k_i}\dot\psi + z_i^{k_i}\varphi_i(z_i)\big) = 0 \tag{4.4.17}$$

for unknown $\varphi_i(z_i)$ defined in a neighborhood of z_i^*. Using *Lemma 1.4.2* we obtain pointwise estimates $|R^{N^*\otimes K_S}_{u_0,J_0}(z_i)| \leqslant C \cdot |z_i|^{k_i}$. Thus equation (4.4.17) is equivalent to

$$\left(\overline\partial^{N^*\otimes K_S}_{u_0,J_0} + \left(\frac{\bar z_i}{z_i}\right)^{k_i} R^{N^*\otimes K_S}_{u_0,J_0}\right)\varphi_i(z_i) + z_i^{-k_i} R^{N^*\otimes K_S}_{u_0,J_0} j^{k_i} = 0. \tag{4.4.18}$$

The existence of solutions of (4.4.18) can be deduced from the surjectivity of the operator $\overline\partial + R : L^{1,p}(\Delta, \mathbb{C}^n) \to L^p(\Delta, \mathbb{C}^n)$ with $R \in L^p$, $p > 2$. We refer to [Iv-Sh-1] for the construction of a right inverse for such $\overline\partial + R$. This implies the local existence of solutions $\varphi_i(z_i)$ of (4.4.17).

The regularity property of $R^{N^*\otimes K_S}_{u_0,J_0}$ implies that the $z_i^{k_i}\varphi_i(z_i)$ are $C^{\ell-1}$-smooth. Thus we can construct a $\dot\psi \in C^{\ell-1}(S, N^*_{u_0} \otimes K_S)$ which locally near z_i^* has the form $\dot\psi(z_i) = j^{k_i}\dot\psi + z_i^{k_i}\varphi_i(z_i)$ and satisfies (4.4.17). Now, the surjectivity of $\Upsilon^\psi_{\boldsymbol{\nu}}$ will follow from the existence of $(v, \dot J_S, \dot J) \in T_{(u_0,J_{S,0},J_0)}\widehat{\mathscr{M}}_{=\boldsymbol{k},h^1=1}$ such that for the constructed $\dot\psi$ and a fixed non-zero $\psi_0 \in \mathsf{H}^0_D(S, N^*_{u_0} \otimes K_S)$ the relation (4.4.14) holds for any $w \in L^{1,p}(S, N_{u_0})$.

Now observe that we can use (4.2.11) to compute $\nabla_{(v,\dot J_v,\dot J)} D^N_{u_0,J_0}$. This implies that we can use the trick from the proof of *Lemma 4.4.1*. Namely, we look for the desired $(v, \dot J_S, \dot J)$ in the special form, such that v and $\dot J_S$ vanish identically, and $\dot J$ vanishes along $u_0(S)$ and in some neighborhoods of cusp-points of $u(S)$. Now all terms in (4.2.11) except [7] vanish, and (4.4.14) is equivalent to

$$D^{N^*\otimes K_S}_{u_0,J_0}\dot\psi + \psi_0 \circ \nabla \dot J \circ du_0 \circ J_S = 0. \tag{4.4.19}$$

To finish the construction of $\dot J$ we use the fact that $D^{N^*\otimes K_S}_{u_0,J_0}\dot\psi$ vanishes in a neighborhood of each cusp-point z_i^*. This yields the surjectivity of $\Upsilon^\psi_{\boldsymbol{\nu}}$ and the first assertion of the lemma.

The second and third assertions follow from previous considerations. □

4.5. **(Non)existence of saddle points in the moduli space.** The results obtained above in this section allow us to prove the main technical result of the paper. Let X be a manifold of dimension $2n$, $\mathscr{J}$ an open connected set in the space of C^ℓ-smooth almost complex structures on X with $\ell > 2$ non-integer, S a closed surface of genus $g \geqslant 1$, and $[C] \in \mathsf{H}_2(X, \mathbb{Z})$ a homology class.

Definition 4.5.1. A pseudoholomorphic map $u : S \to X$ has an *ordinary cusp* at $z^* \in S$ if for appropriate coordinates z on S and (w_1, w_2) on X

$$u(z) = \big(z^2 + O(|z|^3), z^3 + O(|z|^{3+\alpha})\big). \tag{4.5.1}$$

This property is equivalent to the condition that u has a cusp of order 1 and the secondary cusp-index 0 at z^*.

Theorem 4.5.1. *Let $h(t) = J_t$, $t \in I = [0,1]$, be a generic path in $\mathscr{J}$ and $\mathscr{M}_h$ the corresponding relative moduli space of parameterized pseudoholomorphic curves of genus $g \geqslant 1$ in the homology class $[C]$.*

i) If $n \geqslant 3$, then every critical point of the projection $\pi_h : \mathscr{M}_h \to I$ is represented by an imbedded curve $C = u(S)$, $u : S \to X$;

ii) If $n = 2$, then every critical point of the projection $\pi_h : \mathscr{M}_h \to I$ is represented by a curve $C = u(S)$ such that:

- *the only singularities on C are nodes or ordinary cusps;*
- *the possible number of cuspidal points $\varkappa$ on C is*

$$\mu \leqslant \varkappa \leqslant \mu + g - 1, \tag{4.5.2}$$

where $\mu := \langle c_1(X), [C]\rangle$ and g is the (geometric) genus of C, $g = g(S)$;

- *the saddle index of $d^2\pi_h$ at C is at least $\varkappa$, i.e.*

$$\mathsf{S\text{-}ind}_C d^2\pi_h \geqslant \varkappa \geqslant \mu.$$

iii) In the case when the inequality (4.5.2) *is a contradiction, the claim ii) has the following meaning:*

- *If $g = 0$, then π_h has no critical points;*
- *If $\mu + g - 1 < 0$, then the space $\mathscr{M}_h$ is empty for generic h.*

Before giving the proof we must specify the meaning of the notion *generic path*. One of the most reasonable conditions is that any two *regular* almost complex structures J_0, $J_1 \in \mathscr{J}$ (see § *2.3*) can be connected by a path $\{J_t\}_{t \in I=[0,1]}$ with the property stated in the theorem. To ensure this we need the following easy

Proposition 4.5.2. *Let $F : \mathscr{X} \to \mathscr{Y}$ be a C^1-smooth Fredholm map between separable Banach manifolds. Assume that $\mathscr{Y}$ is connected and that the index of F is at most -2. Then the set $\mathscr{Y} \backslash F(\mathscr{X})$ is path-connected.*

Remark. The proposition generalizes the obvious fact that submanifolds of codimension at least 2 do not divide the ambient manifold. Note that one can have at most countably many connected components of $\mathscr{X}$ and that on these components the index of F can vary from component to component.

Proof *Theorem 4.5.1*. We already know from *Section 2* that for a generic path $h(t) = J_t$ in $\mathscr{J}$ the set $\mathscr{M}_h$ is a manifold. In previous paragraphs of this section we have showed that critical points $[u, J]$ of the projection $\pi_h : \mathscr{M}_h \to I$ have an intrinsic description independent of the particular choice of the path J_t. Moreover, the quadratic form $d^2\pi_h$ at these points also admits a similar intrinsic description. Furthermore, we have found a stratification of the set of "suspicious" points $[u, J] \in \mathscr{M}$ by submanifolds and estimated their codimension. It remains to find the strata with the Fredholm index $\leqslant -2$ over $\mathscr{J}$ and apply *Proposition 4.5.2*.

The "suspicious" points $[u, J]$ on $\mathscr{M}$ are those with $\mathsf{H}^1_D(S, \mathscr{N}_u) \cong \mathbb{R}$. They can be separated into classes according to the structure of the normal sheaf $\mathscr{N}_u$. Since the

singular part $\mathscr{N}_u^{\mathsf{sing}}$ of $\mathscr{N}_u$ reflects the cusp-curves we are led to the spaces $\mathscr{M}_{=\boldsymbol{k}}$ of curves with prescribed order of cusps.

Denote by ind the index of the projection $\mathsf{pr}_{\mathscr{J}} : \mathscr{M} \to \mathscr{J}$, so that $\mathsf{ind} = 2(\langle c_1(X), [C]\rangle + (g-1)(3-n))$. If $\mathsf{ind} < 0$, then for a generic path $h(t) = J_t$ the set $\mathscr{M}_h$ is empty and the claim of the theorem holds. Thus we may assume that $\mathsf{ind} \geqslant 0$. By *Theorem 3.2.1*, we must "pay" at least $2(n-1)|\boldsymbol{k}|$ dimensions to go to $\mathscr{M}_{=\boldsymbol{k}}$. By *Lemma 4.4.1*, we must "pay" further $\mathsf{ind} - 2|\boldsymbol{k}| + 1$ dimensions to obtain the condition $\mathsf{H}^1_D(S, \mathscr{N}_u) \cong \mathbb{R}$. Note that $2(n-1)|\boldsymbol{k}| \geqslant 2|\boldsymbol{k}| + 2$ if $n \geqslant 3$ and $\boldsymbol{k}$ is non-trivial. Thus in the case $n \geqslant 3$ we "overdraw" our "credit" ind at least by 3. This means that for non-trivial $\boldsymbol{k}$ the index of the projection from $\mathscr{M}_{=\boldsymbol{k},h^1=1}$ to $\mathscr{J}$ is at most -3 and we can apply *Proposition 4.5.2*. Thus for $n \geqslant 3$ any critical point of $\mathscr{M}_h$ is represented by an immersion $u : S \to X$.

In the case $n = 2$ we can "strike the balance" in a similar way. Indeed, we come to the "overdraw" of at least 3 dimensions in each of the following cases:

a) $\mathsf{H}^1_D(S, \mathscr{N}_u) \cong \mathbb{R}$ and there exists at least one cusp-point of cusp-order$\geqslant 2$;

b) $\mathsf{H}^1_D(S, \mathscr{N}_u) \cong \mathbb{R}$ and there exists at least one cusp-point the secondary cusp-index $\geqslant 1$;

c) $\mathsf{H}^1_D(S, \mathscr{N}_u) \cong \mathbb{R}$ and a non-trivial $\psi \in \mathsf{H}^1_D(S, N_u^* \otimes K_S)$ vanishes in at least one cusp-point.

Thus for generic $h(t) = J_t$ we can exclude all these possibilities. The remaining case admits only cusps of order 1 with the secondary cusp-index 0. This means that u has only ordinary cusps. Since possibility *c)* is excluded, each such cusp gives input 1 into the saddle index by *Lemma 4.3.4*. Finally, we estimate the number of such cusps using *Lemma 2.3.4*.

Now we show that for $n \geqslant 3$ and generic h any critical point of $\mathscr{M}_h$ is represented by an *imbedding* $u : S \to X$. Since this result will be not used in the sequel, we give only a sketchy proof.

Denote by $\mathscr{M}_{\mathsf{imm}}$ the total moduli space of *immersed pseudoholomorphic curves* with the same topological data $g = g(S)$ and $[C] \in \mathsf{H}_2(X, \mathbb{Z})$ as usual. In other words, $\mathscr{M}_{\mathsf{imm}} = \mathscr{M}_{=\boldsymbol{k}}$ with trivial $\boldsymbol{k}$. It follows easily from *Section 3* that this is an open set in the whole space $\mathscr{M}$. The space $\mathscr{M}_{\mathsf{imm}}$ admits a natural stratification in which every stratum contains curves with the same number and type of multiple point on the image $C = u(S)$. Obviously, the biggest stratum is the subspace of *imbedded* curves, and this is an open subset in $\mathscr{M}_{\mathsf{imm}}$. The next biggest stratum consists of curves with exactly one transversal double point on $C = u(S)$. Let us denote it by $\mathscr{M}^{\times}_{\mathsf{imm}}$ with the character $\times$ symbolizing a transversal self-intersection of exactly 2 branches of $C = u(S)$.

Locally, $\mathscr{M}^{\times}_{\mathsf{imm}}$ is defined by the condition $u(z_1) = u(z_2)$ for some $z_1 \neq z_2 \in S$. Linearization of this condition is the equation

$$\mathsf{pr}_{N^\times}(v(z_1) - v(z_2)) = 0$$

on $[v, \dot{J}_S, \dot{J}] \in T_{[u,J]}\mathscr{M}_{\mathsf{imm}}$, where $N^\times$ denotes the plane in $T_{x^\times}X$ normal to both branches of $C = u(S)$ at the point $x^\times = u(z_1) = u(z_2)$, *i.e.* $N^\times := T_{x^\times}X/\big(du(T_{z_1}S) \oplus du(T_{z_2}S)\big)$. It is easy to see that this condition is transversal. Thus $\mathscr{M}^{\times}_{\mathsf{imm}}$ is a C^ℓ-smooth submanifold of real codimension $2(n-2)$. Moreover, it follows from the proof of *Lemma 4.4.1* that biggest stratum is transversal to the subspace $\mathscr{M}_{\mathsf{imm},h^1=1}$ of immersed curves with $\mathsf{h}^1(S, \mathscr{N}_u) = 1$. Consequently, the space $\mathscr{M}^{\times}_{\mathsf{imm},h^1=1} := \mathscr{M}^{\times}_{\mathsf{imm}} \cap \mathscr{M}_{\mathsf{imm},h^1=1}$ of immersed curves with $\mathsf{h}^1(S, \mathscr{N}_u) = 1$ and with exactly one transversal self-intersection point is a C^ℓ-smooth

submanifold of real codimension $2(n-2)$ in $\mathscr{M}_{\mathsf{imm},h^1=1}$, and of real codimension $\mathsf{ind}+1+2(n-2)$ in $\mathscr{M}$. Since $2(n-2)\geqslant 2$ for $n\geqslant 3$, we can apply *Proposition 4.5.2*.

The complementary strata $\mathscr{M}^{\boldsymbol{a}}_{\mathsf{imm}}$ of $\mathscr{M}_{\mathsf{imm}}$ consist of curves having either several double points, or one double point with tangency of higher degree, or even more complicated multiple points, with the index $\boldsymbol{a}$ encoding the number and the type of multiple points. A similar argument shows that these strata $\mathscr{M}^{\boldsymbol{a}}_{\mathsf{imm}}$ are transversal to $\mathscr{M}_{\mathsf{imm},h^1=1}$ and that the intersections $\mathscr{M}^{\boldsymbol{a}}_{\mathsf{imm},h^1=1}:=\mathscr{M}^{\boldsymbol{a}}_{\mathsf{imm}}\cap\mathscr{M}_{\mathsf{imm},h^1=1}$ are transversal. The computation of the number of conditions shows that these strata have even higher codimension in $\mathscr{M}_{\mathsf{imm}}$. So *Proposition 4.5.2* still applies. This finishes the proof of the theorem. □

In applications, one needs a version of *Theorem 4.5.1* for the case of curves passing through given fixed points $\boldsymbol{x}=(x_1,\dots,x_m)$ on X. Recall that for a C^ℓ-smooth map $h:I:=[0,1]\to\mathscr{J}$ we denote by $\mathscr{M}_{h,\boldsymbol{x}}$ the relative moduli space of $J_t=h(t)$-holomorphic curves passing through $\boldsymbol{x}=(x_1,\dots,x_m)$ (see *Paragraph 2.4*). Let $\pi_{h,\boldsymbol{x}}:\mathscr{M}_{h,\boldsymbol{x}}\to I$ be the corresponding projection. We also assume that $\mathsf{dim}_{\mathbb{R}}X=4$.

Theorem 4.5.3. *For a generic h every critical point of the projection $\pi_{h,\boldsymbol{x}}:\mathscr{M}_{h,\boldsymbol{x}}\to I$ is represented by a curve C such that:*

- *the only singularities on C are nodes or ordinary cusps;*
- *the marked points $(x_1,\dots,x_m)$ are smooth points of $C=u(S)$;*
- *the possible number of cuspidal points $\varkappa$ on C is*
$$\mu-m\leqslant\varkappa\leqslant\mu-m+g-1 \tag{4.5.3}$$
where g is the (geometric) genus of C;
- *the saddle index of $d^2\pi_h$ at C is at least $\varkappa$,* i.e.
$$\mathsf{S\text{-}ind}_C d^2\pi_h\geqslant\varkappa\geqslant\mu-m.$$

In the case when the inequality (4.5.3) *is a contradiction the claim has the following meaning:*

- *If $g=0$, then $\pi_{h,\boldsymbol{x}}$ has no critical points;*
- *If $\mu-m+g-1<0$, then the space $\mathscr{M}_h$ is empty for generic h.*

Proof. The main observation in the proof is that after an appropriate modification all the results of this section remain valid also for curves passing through fixed points. In particular, the most important formulas (4.3.7) and (4.3.9) from *Lemma 4.3.3* holds after replacing $\mathscr{N}^{\mathsf{sing}}_u$ by $\mathscr{N}^{\mathsf{sing}}_{u,\boldsymbol{x}}$. To show this we note first that *Lemmas 4.2.3, 4.3.1*, and *4.3.2* can be applied without any modification. After this, the proof of *Lemma 4.3.3* applies with the only difference that the usual Gromov operator $D_{u,J}$ acting in E should be replaced by the operator $D_{u,-\boldsymbol{z},J}$ acting in $E_{-\boldsymbol{z}}$. The validation of such a replacement is justified in *Paragraph 2.4*. Indeed, by the very definition, $D_{u,-\boldsymbol{z},J}$ is the restriction of $D_{u,J}$ to the subspace of sections of the subbundle $E_{u,-\boldsymbol{z}}\subset E_u$. In a similar way one modifies the argumentation of *Paragraph 4.4*.

Finally, we note that the condition of coincidence of some cusp point of $C=u(S)$ with some of marked points $x_1,\dots,x_m$ defines a subset in $\mathscr{M}_{\boldsymbol{x}}$ which has a natural stratification into submanifolds of codimension $\geqslant 2$. Every such stratum is defined by the cusp order $\boldsymbol{k}$ of $C=u(S)$ and indication of the those cuspidal points which pass through the marked points $x_1,\dots,x_m$. This means that every such stratum is a submanifold of the space $\mathscr{M}_{=\boldsymbol{k}}$. Moreover, the codimension of every such stratum in $\mathscr{M}_{=\boldsymbol{k}}$ is $4a$, where a is the number of cusps lying in the marked points. As in the case $m=0$ above, one can show that the intersection of such a stratum with the space $\mathscr{M}_{=\boldsymbol{k},h^1=1}$ is transversal and has the expected

codimension. Hence we may conclude that for generic h such a coincidence can not occur in the critical points of $\pi_{h,\boldsymbol{x}}$. The same argument is applied to show that for generic h there are no coincidence of the marked points $x_1, \dots, x_m$ with nodal points of the curve $C = u(S)$ representing a critical point of π_h. □

5. Deformation of nodal curves

5.1. **Nodal curves and Gromov compactness theorem.** The total moduli space $\mathscr{M}$ constructed in *Section 2* is not complete. More precisely, the projection $\pi_{\mathscr{J}} : \mathscr{M} \to \mathscr{J}$ is, in general, not proper. This means that there exists a sequence $[u_i, J_i] \in \mathscr{M}$ such that J_i converges to $J_\infty \in \mathscr{J}$ but no subsequence of $\{u_i\}$ converges in $L^{1,p}(S, X)$-topology, even after reparameterization. Gromov compactness theorem ensures that there still exists subsequence of $\{u_i\}$ which converges with respect to the Gromov topology, which is weaker that the Sobolev $L^{1,p}$-topology.

In the literature one can find several non-equivalent definitions for Gromov topology. In this paper we shall use that one which is equivalent to the original definition of Gromov ([Gro]). However, our version is more detailed in the sense that it is based on the notion of *stable maps*. This notion for curves in a complex algebraic manifold X was introduced by Kontsevich in [K], see also [K-M]. Our definition of stable maps over (X, J) is simply a translation of this notion to almost complex manifolds.

Definition 5.1.1. The *standard node* is the complex analytic set

$$\mathscr{A}_0 := \{(z_1, z_2) \in \Delta^2 : z_1 \cdot z_2 = 0\}. \tag{5.1.1}$$

A point on a complex curve is called a *nodal point*, if has a neighborhood biholomorphic to the standard node. A *nodal curve* C is a complex analytic space of pure dimension 1 with only nodal points as singularities.

Definition 5.1.2. An annulus A with a complex structure J has *conformal radius* $R > 1$ if A is biholomorphic to $A(1, R) := \{z \in \mathbb{C} : 1 < |z| < R\}$. Define a *cylinder* $Z(a, b) := S^1 \times [a, b] = \{(\theta, t) : 0 \leqslant \theta \leqslant 2\pi,\ a \leqslant t \leqslant b\}$, $a < b$, with the complex structure $J_Z(\frac{\partial}{\partial\theta}) := \frac{\partial}{\partial t}$. Obviously, $Z(a, b)$ is also an annulus A of conformal radius $R = e^{b-a}$. Also denote $Z_k := Z(k, k+1)$.

In other terminology, nodal curves are called *prestable*. We shall always suppose that C is connected and has a "finite topology", *i.e.* C has finitely many irreducible components, finitely many nodal points, and that C has a smooth boundary ∂C consisting of finitely many smooth circles γ_i, such that $\overline{C} := C \cup \partial C$ is compact.

Definition 5.1.3. A real oriented surface with boundary $(\Sigma, \partial\Sigma)$ *parameterizes* a complex nodal curve C if there is a continuous map $\sigma : \overline{\Sigma} \to \overline{C}$ such that:

i) if $a \in C$ is a nodal point, then $\gamma_a = \sigma^{-1}(a)$ is a smooth imbedded circle in $\Sigma \backslash \partial\Sigma$, and if $a \neq b$ then $\gamma_a \cap \gamma_b = \varnothing$;

ii) $\sigma : \overline{\Sigma} \backslash \bigcup_{i=1}^N \gamma_{a_i} \to \overline{C} \backslash \{a_1, \dots, a_N\}$ is a diffeomorphism, where $a_1, \dots, a_N$ are the nodes of C.

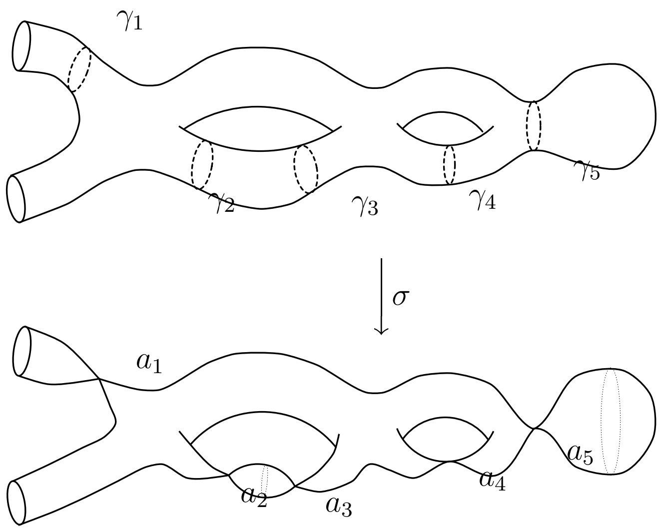

Fig. 1

Circles $\gamma_1, ..., \gamma_5$ are contracted by the parameterization map σ to nodal points $a_1, \dots a_5$.

Note that such a parameterization is not unique: if $g : \overline{\Sigma} \to \overline{\Sigma}$ is any orientation preserving diffeomorphism then $\sigma \circ g : \overline{\Sigma} \to \overline{C}$ is again a parameterization.

A parameterization of a nodal curve C by a real surface can be considered as a method of "smoothing" of C. We shall also use an alternative method of "smoothing", the normalization. Consider the normalization $\hat{C}$ of C. Mark on each component of this normalization the pre-images (under the normalization map $\pi_C : \hat{C} \to C$) of nodal points of C. Let $\hat{C}_i$ be a component of $\hat{C}$. We can also obtain $\hat{C}_i$ by taking an appropriate irreducible component C_i, replacing nodes contained in C_i by pairs of discs with marked points, and marking the remaining nodal points. Since it is convenient to consider components in this form, we make the following

Definition 5.1.4. A *component* C' of a nodal curve C is the normalization of an irreducible component of C with marked points selected as above.

This definition allows us to introduce Sobolev and Hölder spaces of functions and (continuous) maps of nodal curves.

Definition 5.1.5. A continuous map $u : C \to X$ is Sobolev $L^{1,p}$-smooth, $u \in L^{1,p}(C, X)$ if the induced maps $u_i := u|_{C_i} : C_i \to X$ of all of its components C_i are $L^{1,p}$-smooth. The notion of J-holomorphic maps $u : C \to X$ is similarly defined. For $u \in L^{1,p}(C, X)$ define $E_u := u^*TX$. Thus E_u is determined by restricting to each component and by identifying fibers over pairs (z', z'') of marked points corresponding to nodal points. An $L^{1,p}$-smooth section v of E_u over C is given by a collection of sections $v_{C_i} \in L^{1,p}(C_i, E_u)$, one for every component C_i of C, such that $v(z') = v(z'')$ for each pair (z', z'') of marked points chosen as above. Denote by $L^{1,p}(C, E_u)$ the space of $L^{1,p}$-sections of E_u.

Definition 5.1.6. The *energy* or the *area* of a continuous $L^{1,2}_{\mathsf{loc}}$-smooth map with respect to a metric h on X is defined as

$$\mathsf{area}_h(u) := \|du\|^2_{L^2(C)} = \int_C |du|^2_h \tag{5.1.2}$$

This definition depends only on the complex structure on C but not on the choice of a metric on C in the given conformal class. If an ω-tame almost complex structure J is given, there is a prefered choice of a metric h on X defined by $h(v, v) := \omega(v, Jv)$ for $v \in TX$.

Remark. Our definition of the area uses the following fact. Let g be a Riemannian metric on C compatible with j_C, h a Riemannian metric on X, and $u : C \to X$ a J-holomorphic immersion. Then $\|du\|^2_{L^2(C)}$ is independent of the choice of g and coincides with the area of the image $u(C)$ with respect to the metric $h_J(\cdot,\cdot) := \frac{1}{2}(h(\cdot,\cdot) + h(J\cdot, J\cdot))$. The metric h_J here can be seen as a "Hermitization" of h with respect to J. It is well-known that $\|du\|^2_{L^2(C)}$ is independent of the choice of a metric g on C in the same conformal class, see *e.g.* [S-U]. Thus we can use the flat metric $dx^2 + dy^2$ to compare area and energy. For a J-holomorphic map we obtain

$$\|du\|^2_{L^2(C)} = \int_C |\partial_x u|^2_h + |\partial_y u|^2_h = \int_C |\partial_x u|^2_h + |J\partial_x u|^2_h = \int_C |du|^2_{h_J} = \mathsf{area}_{h_J}(u(C)), \quad (5.1.3)$$

where the last equality is another well-known result, see *e.g.* [Gro]. Since we consider varying almost complex structures on X, it is useful to know that we can use any Riemannian metric on X having a reasonable notion of area.

Definition 5.1.7. A *stable curve over* (X, J) is a pair (C, u), where C is a nodal curve and $u : C \to X$ is a J-holomorphic map satisfying the following condition: If C' is a closed component of C such that u is constant on C', then there exist only finitely many biholomorphisms of C' which preserve the marked points of C. In this case u is called a *stable map*.

Remark. One can see that stability condition is nontrivial only in the following cases:

1) some component C' is biholomorphic to $\mathbb{CP}^1$ with 1 or 2 marked points; in this case u should be non-constant on any such component C';

2) some irreducible component C' of C is $\mathbb{CP}^1$ or a torus without nodal points.

Since we consider only connected nodal curves, case *2)* can occur only if C irreducible, *i.e.* $C' = C$. In this case u must be non-constant on C.

Definition 5.1.8. A component C' of a nodal curve C is called *non-stable* in the following cases:

1) C' is $\mathbb{CP}^1$ and has one or two marked points;

2) C' is $\mathbb{CP}^1$ or a torus and has no marked points.

Let $u : C \to X$ be a pseudoholomorphic map. An irreducible component C' of C is a *ghost* component (with respect to u) if u is constant on C'. The *ghost part* C^{gh} *of* C *(with respect to* u*)* is the union of all ghost components. In this paper we shall deal only with the case when all ghost components are closed.

A map u is *non-multiple* if, except finitely many points $z \in C$, one has $u^{-1}(u(z)) = \{z\}$. Note that this condition excludes also ghost components.

Now we are going to describe the Gromov topology on the space of stable curves over X introduced in [Gro]. Let $\{J_n\}$ be a sequence of continuous almost complex structures on X which converges to J_∞ in the C^0-topology. Furthermore, let (C_n, u_n) be a sequence of stable curves over (X, J_n), such that all C_n are parameterized by the same real surface S.

Definition 5.1.9. We say that (C_n, u_n) *converges in the Gromov topology to a stable* J_∞*-holomorphic curve* (C_∞, u_∞) *over* X if the parameterizations $\sigma_n : \overline{S} \to \overline{C}_n$ and $\sigma_\infty : \overline{S} \to \overline{C}_\infty$ can be chosen in such a way that the following holds:

i) $u_n \circ \sigma_n$ converges to $u_\infty \circ \sigma_\infty$ in the $C^0(S, X)$-topology;

ii) if $\{a_k\}$ is the set of nodes of C_∞ and $\{\gamma_k\}$ are the corresponding circles in S, then on any compact subset $K \Subset S \backslash \cup_k \gamma_k$ the convergence $u_n \circ \sigma_n \to u_\infty \circ \sigma_\infty$ is $L^{1,p}(K, X)$ for all $p < \infty$;

iii) for any compact subset $K \Subset \overline{S} \backslash \cup_k \gamma_k$ there exists $n_0 = n_0(K)$ such that $\sigma_n^{-1}(\{a_k\}) \cap K = \varnothing$ for all $n \geqslant n_0$ and the complex structures $\sigma_n^* j_{C_n}$ converge smoothly to $\sigma_\infty^* j_{C_\infty}$ on K;

iv) the structures $\sigma_n^* j_{C_n}$ are independent of n near the boundary ∂S.

Condition *iv)* is trivial if S is closed, but it is useful when one considers the "free boundary case", *i.e.* when S (and thus all C_n) are not closed and no boundary condition is imposed.

The reason for introducing the notion of a curve stable over X is similar to the one for the Gromov topology. We are looking for a completion of the space of smooth imbedded pseudoholomorphic curves which has "nice" properties, namely: 1) such a completion should contain the limit of a subsequence of every sequence of smooth curves which is bounded in an appropriate sense; 2) such a limit should also exist for every sequence in the completed space; 3) such a limit should be unique. The Gromov's compactness theorem ensures us that the space of curves stable over X has these nice properties.

Definition 5.1.10. Let C_n be a sequence of nodal curves, parameterized by the same real surface S. We say that the complex structures on C_n *do not degenerate near boundary*, if there exist $R > 1$, such that for any n and any boundary circle $\gamma_{n,i}$ of C_n there exist an annulus $A_{n,i} \subset C_n$ adjacent to $\gamma_{n,i}$, such that all $A_{n,i}$ are mutually disjoint, do not contain nodal points of C_n, and have the same conformal radius R.

Since the conformal radii of all $A_{n,i}$ are all the same, we can identify them with $A(1, R)$. This means that all changes of complex structures of C_n take place away from boundary. The condition is trivial if C_n and S are closed, $\partial S = \partial C_n = \varnothing$.

Remark. Changing our parameterizations $\sigma_n : S \to C_n$, we may suppose that for any i the pre-image $\sigma_n^{-1}(A_{n,i})$ is the same annulus A_i independent of n.

Now we state Gromov's compactness theorem for stable curves. Assume that X is a compact manifold and fix some Riemannian metric h on X.

Theorem 5.1.1. *Let (C_n, u_n) be a sequence of stable J_n-holomorphic curves over X with parameterizations $\delta_n : S \to C_n$. Suppose that:*

- *a) $\{J_n\}$ is a sequence of continuous almost complex structures on X, which converges to J_∞ in the C^0-topology;*
- *b) there is a constant M such that $\mathsf{area}_h[u_n(C_n)] \leqslant M$ for all n;*
- *c) complex structures on the C_n do not degenerate near the boundary.*

Then there is a subsequence (C_{n_k}, u_{n_k}) and parameterizations $\sigma_{n_k} : S \to C_{n_k}$, such that $(C_{n_k}, u_{n_k}, \sigma_{n_k})$ converges to a J_∞-holomorphic curve $(C_\infty, u_\infty, \sigma_\infty)$ stable over X.

Moreover, the limit curve (C_{n_k}, u_{n_k}) is unique up to the choice of the parameterization σ_∞.

Furthermore, if the structures $\delta_n^* j_{C_n}$ are constant on the fixed annuli A_i, each adjacent to a boundary circle γ_i of S, then the new parameterizations σ_{n_k} can be taken equal to δ_{n_k} on some subannuli $A'_i \subset A_i$, also adjacent to γ_i.

A detailed proof of the theorem in the stated form can be found in [Iv-Sh-3]. We also refer to the original proof of Gromov in [Gro].

Gromov's compactness theorem induce a natural completion of the moduli space $\mathscr{M}$. Let a closed real surface S of genus g and a homology class $A \in \mathsf{H}_2(X,\mathbb{Z})$ be given.

Definition 5.1.11. Nodal J-holomorphic curves $u' : C' \to X$ and $u'' : C'' \to X$ are *equivalent* if there exists a biholomorphism $\varphi : C' \to C''$ with $u' = u'' \circ \varphi$. The *total moduli space* $\overline{\mathscr{M}}^{st}$ *of stable nodal curves over* X is the set of equivalence classes $[C,u,J]$ with $J \in \mathscr{J}$ and $u : C \to X$ a stable J-holomorphic curve representing a given class $A \in \mathsf{H}_2(X,\mathbb{Z})$. The space $\overline{\mathscr{M}}^{st}$ is equipped with the *Gromov topology* in which a sequence $[C_n, u_n, J_n]$ converges to $[C_\infty, u_\infty, J_\infty]$ if J_n converges to J_∞ in the C^ℓ-topology and (C_n,u_n) to $(C_\infty, u_\infty,)$ in the sense of *Definition 5.1.9*. Denote by $\mathsf{pr}^{st}_{\mathscr{J}}$ the natural projection $\mathsf{pr}^{st}_{\mathscr{J}} : [C,u,J] \in \overline{\mathscr{M}}^{Gr} \mapsto J \in \mathscr{J}$.

Define the *Gromov compactification* $\overline{\mathscr{M}}^{Gr}$ of the total moduli space $\mathscr{M}$ of pseudoholomorphic curves X as the closure of $\mathscr{M}$ in $\overline{\mathscr{M}}^{st}$.

Note that every fiber $\overline{\mathscr{M}}^{Gr}_J := \overline{\mathscr{M}}^{Gr} \cap \big(\mathsf{pr}^{st}_{\mathscr{J}}\big)^{-1}(J)$ is compact. Note also that in general $\overline{\mathscr{M}}^{st} \neq \overline{\mathscr{M}}^{Gr}$. This means that there are stable curves $[C,u,J] \in \overline{\mathscr{M}}^{st}$ which can not be reached from $\mathscr{M}$.

5.2. **The cycle topology for pseudoholomorphic curves.** The Gromov compactness theorem gives a precise description of the behavior of *parameterized* pseudoholomorphic curves at "infinity" of the total moduli space. However, what we are really interested in is not a pseudoholomorphic map $u : C \to X$ itself but rather the image $u(C) \subset X$, *i.e.* a *non-parameterized* pseudoholomorphic curve. The natural space where non-parameterized curves "live" is the space $\mathscr{Z}_2(X)$ of 2-currents on the ambient manifold X. Recall that $\mathscr{Z}_2(X)$ is the dual space to the space $C^\infty(X, \Lambda^2 X)$ of smooth 2-forms on X (see *e.g.* [Gr-Ha], *Chapter 3*).

Definition 5.2.1. Let X be a manifold, C an abstract nodal curve with the smooth boundary ∂C such that $\overline{C} := C \cup \partial C$ is compact, and $u : C \to X$ a map which is $L^{1,p}$-smooth up to boundary. Define the *cycle* $u[C]$ *associated with the map* $u : C \to X$ as the current whose pairing with a smooth 2-form φ on X equals $\langle u(C), \varphi \rangle := \int_C u^*\varphi$. In this case we also say that $u[C]$ is *represented by the map* $u : C \to X$.

If additionally $u : C \to X$ is J-holomorphic with respect to some an almost complex structure on X, we call $C' := u[C]$ an *J-holomorphic curve C in X*. In this case we say that the J-holomorphic curve (C,u) over X and the map $u : C \to X$ *represent the curve* $C' = u[C]$. The set $u(C)$ is called the *support of* C' and denoted by $\mathsf{supp}\,(C')$.

A curve C' in X is *non-multiple* if it can be represented by a non-multiple pseudoholomorphic map $u : C \to X$ (see *Definitions 1.2.2* and *5.1.8*). In this case we identify the set $u(C)$ and the current $u[C]$ and use the same notation $u(C)$.

A sequence of cycles $u_n[C_n]$ *converges to a cycle* $u_\infty[C_\infty]$ if $\langle u_n(C_n), \varphi \rangle$ converges to $\langle u_\infty(C_\infty), \varphi \rangle$ for any smooth smooth 2-form φ on X. In other words, the cycle topology is the topology induced from the space of currents $\mathscr{Z}_2(X)$.

Lemma 5.2.1. *i) Let (X,J) be an almost complex manifold and (C,u), (C',u') closed J-holomorphic curves over X. Assume that*

- *C and C' are parameterized by the same closed surface S;*
- *(C,u) contains no multiple and ghost components;*
- *the associated cycles $u[C]$ and $u'[C']$ coincide.*

Then (C,u) *are* (C',u') *equivalent.*

ii) Let J_n *be a sequence of continuous almost complex structures on* X *which converges to an almost complex structure* J_∞ *in the* C^0*-topology, and* (C_n,u_n) *a sequence of stable* J_n*-holomorphic closed curves over* X *which converges to* (C_∞,u_∞) *in the Gromov topology. Then* $u_n[C_n]$ *converges to* $u_\infty[C_\infty]$ *in the cycle topology.*

iii) Let J_n *be a sequence of continuous almost complex structures on* X *which converges to an almost complex structure* J_∞ *in the* C^0*-topology,* (C_n,u_n) *a sequence of stable* J_n*-holomorphic closed curves over* X*, and* (C_∞,u_∞) *a parameterized* J_∞*-holomorphic curve. Assume that*

- $u_n[C_n]$ *converges to* $u_\infty[C_\infty]$ *in the cycle topology;*
- C_n *and* C_∞ *are parameterized by the same closed surface* S*;*
- (C_∞,u_∞) *contains no multiple and ghost components;*
- J_∞ *is* C^1*-smooth.*

Then (C_n,u_n) *converges to* (C_∞,u_∞) *in the Gromov topology.*

Proof. *Part i).* The hypotheses on (C,u) and (C',u') imply that (C',u') also contains no multiple and ghost components. The claim then follows from the unique continuation property of pseudoholomorphic curves (see *Lemma 1.2.5*).

Part ii). This follows from the definition of the Gromov topology and the description of the convergence at nodes given in *Step 0)* of the proof of *Lemma 5.3.1* below.

Part iii). Fix a J_∞-Hermitian metric h on X. Let ω be the associated 2-form, $\omega(v,w) := h(J_\infty v,w)$. Then the structures J_n are ω-tame for $n \gg 1$. Note that even if ω is apriori only continuous and not closed, the notion of ω-tameness is still meaningful. Moreover, the ω-tameness provides a uniform bound of h-area of $u_n[C_n]$. Consequently, some subsequence $(C_{n'},u_{n'})$ of (C_n,u_n) converges in the Gromov topology to a stable J_∞-holomorphic curve (C'_∞,u'_∞). The hypotheses of the corollary imply that (C'_∞,u'_∞) is equivalent to (C_∞,u_∞) and the result follows. □

Remark. The meaning of *Lemma 5.2.1* is that, in the absence of multiple and ghost components, the notions of pseudoholomorphic curves *over* X and *in* X essentially coincide. The same also holds for the Gromov and the cycle topologies. Note also that several authors ([Ye], [Pa-Wo], [Hum]) considered a weaker version of the Gromov compactness theorem where the cycle topology is used instead of the Gromov one.

Definition 5.2.2. Define the *cycle compactification* $\overline{\mathscr{M}}$ of the total moduli space $\mathscr{M}$ as the set of pairs (C,J), where $J \in \mathscr{J}$ and C is a J-holomorphic curve *in* X which considered as the cycle $u[C']$ represented by some J-holomorphic map $u: C' \to X$. Equip the space $\overline{\mathscr{M}}$ with the *cycle topology* in which a sequence (C_n,J_n) converges to (C_∞,J_∞) if J_n converges to J_∞ in the C^ℓ-topology and C_n converges to C_∞ in the sense of *Definition 5.1.9*. Denote by $\mathsf{pr}_{\mathscr{J}}$ the natural projection $\mathsf{pr}_{\mathscr{J}} : (C,J) \in \overline{\mathscr{M}} \mapsto J \in \mathscr{J}$. Define the natural projection $\mathsf{pr}^{Gr} : \overline{\mathscr{M}}^{Gr} \to \overline{\mathscr{M}}$ by $\mathsf{pr}^{Gr} : (C,u,J) \in \overline{\mathscr{M}}^{Gr} \mapsto u[C] \in \overline{\mathscr{M}}$.

Definition 5.2.3. A *normal parameterization* of a J-holomorphic curve C in X is given by a Riemann surface S (possibly not connected) and a map $u: S \to X$ such that

1) $u[S] = C$;
2) u is J-holomorphic and $L^{1,p}$-smooth up to boundary;
3) the restriction of u to every connected component of S is non-multiple; in particular, there are no ghost components, *i.e.* u is non-constant on every connected component of S;

4) the number of boundary circles of S is as small as possible.

Remark. Without condition (3) one could add new ghost spheres to S and make the Euler characteristic $\chi(S)$ arbitrarily large. Condition (4) excludes the possibility of dividing components of S into pieces which also allows to increase $\chi(S)$.

Lemma 5.2.2. *Let J be a C^1-smooth almost complex structure on X, C an abstract nodal curve, and $u : C \to X$ a J-holomorphic map which is an imbedding near the boundary ∂C. Then, up to a diffeomorphism, there exists a unique normal parameterization $\tilde{u} : S \to X$ of $u[C]$.*

Proof. Let $C = \cup_i C_i$ be the decomposition of C into irreducible components. Denote by m_i the degree u on C_i. This means that

- $m_i = 0$ if C_i is a ghost component, *i.e.* u is constant on C_i;
- m_i is the number of points in the preimage $u^{-1}(x)$ for a generic $x \in u(C_i)$ otherwise.

For every non-zero m_i, denote by S_i the normalization of the image $u(C_i)$. Denote by $\tilde{u}_i : S_i \to u(C_i)$ the corresponding normalization maps. In particular, $m_i = 1$ for every non-closed component C_i and in this case S_i is the normalization of C_i. Define S as the disjoint union of the surfaces S_i, each taken m_i times. Let $\tilde{u} : S \to X$ be the map which coincides with the composition $\tilde{u}_i : S_i \to u(C_i) \hookrightarrow X$ on every copy of S_i. One can see that $\tilde{u} : S \to X$ is a normal parameterization of $u[C]$.

The uniqueness of such a normal parameterization follows from *Lemma 1.2.5* □

Remark. Let us give an example showing that the condition on the behavior of u at the boundary imposed in *Lemma 5.2.2* is necessary. Define curves C' and C'' as the disjoint unions $C' := \{z \in \mathbb{C} : |z| < 2\} \sqcup \{z \in \mathbb{C} : 1 < |z| < 3\}$ and $C'' := \{z \in \mathbb{C} : |z| < 3\} \sqcup \{z \in \mathbb{C} : 1 < |z| < 2\}$. Let $u' : C' \to \mathbb{C}$ and $u'' : C'' \to \mathbb{C}$ be the maps which are the standard imbeddings on every component of C' and C''. Then obviously (C', u') and (C'', u'') are not equivalent in the sense of *Definition 5.1.11* and define non-equivalent normal parameterizations of $u'[C'] = u''[C'']$.

Corollary 5.2.3. *Under the hypotheses of* Lemma 5.2.2, *the curve $u[C]$, considered as a current $u[C] \in \mathscr{Z}_2(X)$, admits a unique representation in the form*

$$u[C] = \textstyle\sum_i m_i u_i[C_i],$$

where the $u_i : C_i \to X$ are J-holomorphic maps and the $u_i(C_i)$ are the irreducible components of $\mathsf{supp}\,(u[C])$.

The corollary ensures that the notions of an irreducible component and the multiplicity of a closed pseudoholomorphic curve *in* X are well-defined.

The importance of the notion of a normal parameterization lies in the fact that it allows us to define a natural stratification of the cycle compactification $\overline{\mathscr{M}}$ of the total moduli space. Let S be a given connected real surface S of genus g and $[C] \in \mathsf{H}_2(X, \mathbb{Z})$ a homology class, and $\overline{\mathscr{M}} = \overline{\mathscr{M}}(S, X, [C])$ the cycle compactification of the total space $\mathscr{M} = \mathscr{M}(S, X, [C])$ of irreducible pseudoholomorphic curves of genus g in the homology class $[C]$. Take $(C, J) \in \mathscr{M}(S, X, [C])$ and consider a normal parameterization $u' : S' \to X$ of C. Let $C = \sum_i m_i C_i$ be the decomposition into irreducible components in the sense of *Corollary 5.2.3*. Restricting u' to appropriate connected components we obtain normal parameterizations $u'_i : S'_i \to X$ of the corresponding C_i.

Definition 5.2.4. The *topological type of a component* C_i is the triple $(S_i', m_i, [C_i])$, where $[C_i]$ denotes the homology class of C_i. The *topological type* $\boldsymbol{\tau}$ *of a curve* $(C, J) \in \mathscr{M}(S, X, [C])$ is the sequence of all topological types of components $(S_i', m_i, [C_i])$ defined up to permutation.

Lemma 5.2.4. *i) The space* $\mathscr{M}_{\boldsymbol{\tau}}$ *of pseudoholomorphic curves* $(C, J) \in \overline{\mathscr{M}}(S, X, [C])$ *of a given topological type* τ *is a* C^ℓ*-smooth Banach manifold. The natural projection* $\mathsf{pr}_{\mathscr{J}} : \mathscr{M}_{\boldsymbol{\tau}} \to \mathscr{J}$ *is a* C^ℓ*-smooth Fredholm map.*

ii) The space $\overline{\mathscr{M}} = \overline{\mathscr{M}}(S, X, [C])$ *is the union of subspaces* $\mathscr{M}_{\boldsymbol{\tau}}$*.*

Proof. The decomposition $C = \sum_i m_i C_i$ of every $(C, J) \in \mathscr{M}_{\boldsymbol{\tau}}$ shows that $\mathscr{M}_{\boldsymbol{\tau}}$ is the fiber product of the spaces $\mathscr{M}(S_i', X, [C_i])$ over all triples $(S_i', m_i, [C_i]) \in \boldsymbol{\tau}$ taken over the space $\mathscr{J}$,

$$\mathscr{M}_{\boldsymbol{\tau}} = \textstyle\prod_{(S_i', m_i, [C_i]) \in \boldsymbol{\tau}} \mathscr{M}(S_i', X, [C_i]) \,\big/ \mathscr{J}.$$

Checking the transversality condition, one obtains the desired differentiable structure on $\mathscr{M}_{\boldsymbol{\tau}}$.

The second assertion of the lemma is obvious. □

5.3. **Fine apriori estimates for convergence at a node.** For the purpose of this paper we need a refined version of the *Second apriori estimate* given in [Iv-Sh-3], *Lemma 3.4*. This gives a precise description with estimates of the Gromov convergence in neighborhoods of the contracted circles.

Lemma 5.3.1. *Let* X *be a compact manifold* X*,* J^* *a* $C^{0,s}$*-smooth almost complex structure on* X *with* $s > 0$*, and* h *a metric on* X*. Then there exist constants* $\varepsilon = \varepsilon(X, h, J^*, s) > 0$ *and* $C < \infty$ *such that for any* $C^{0,s}$*-smooth almost complex structure* J *with*

$$\|J - J^*\|_{C^{0,s}(X)} \leqslant \varepsilon \tag{5.3.1}$$

and any J*-holomorphic map* $u : Z(0, l) \to X$ *the condition*

$$\|du\|_{L^2(Z_k)} \leqslant \varepsilon \quad \text{for any } k \in [0, l-1] \tag{5.3.2}$$

implies the uniform estimate

$$\|du\|^2_{L^2(Z_k)} \leqslant C \cdot e^{-2k} \cdot \|du\|^2_{L^2(Z(0,2))} + C \cdot e^{-2(l-k)} \cdot \|du\|^2_{L^2(Z(l-2,l))} \tag{5.3.3}$$

for any $k \in [1, l-2]$*.*

Proof. *Step 0). Lemma 3.3* in [Iv-Sh-3] states that under hypotheses of the lemma one has a "local" estimate

$$\|du\|^2_{L^2(Z_k)} \leqslant \frac{\gamma}{2} \left(\|du\|^2_{L^2(Z_{k-1})} + \|du\|^2_{L^2(Z_{k+1})} \right) \qquad \text{for any } k \in [1, l-2] \tag{5.3.4}$$

with a universal constant $\gamma < 1$. Then in *Corollary 3.4* in [Iv-Sh-3] it is shown that (5.3.4) implies the estimate

$$\|du\|^2_{L^2(Z_k)} \leqslant e^{-2\alpha(k-1)} \cdot \|du\|^2_{L^2(Z(0,2))} + e^{-2\alpha(l-2-k)} \cdot \|du\|^2_{L^2(Z(l-2,l))} \tag{5.3.5}$$

for any $k \in [1, l-2]$ with a constant $\alpha > 0$ related to γ by $\gamma = \frac{1}{\cosh(2\alpha)}$.

Remark. Note that in the proof of the estimates (5.3.4) and (5.3.5) are proven in [Iv-Sh-3] under the following assumption: It is supposed that J^* and J in question are only continuous and that $\|J - J^*\|_{C^0(X)} \leqslant \varepsilon'$ for some $\varepsilon' = \varepsilon'(X, J^*, h) > 0$ independent of J.

From the relation $\gamma = \frac{1}{\cosh(2\alpha)}$ we see that the smaller the γ we have the bigger the α in (5.3.5) we obtain. For our purpose it would be sufficient to prove estimate (5.3.4) with the parameter $\gamma^* := \frac{1}{\cosh 2}$. Note however that in the "ideal" case when (X, J, h) is $\mathbb{C}^n$ with the standard complex and Hermitian structures, γ^* is exactly the best possible constant, see the proof of *Lemma 3.3* in [Iv-Sh-3]. Thus one can not expect that estimate (5.3.4) holds with *uniform* $\gamma \leqslant \gamma^*$. The idea is to consider (5.3.4) with parameters γ_k depending on k and to estimate the difference $\gamma_k - \gamma^*$.

Step 1). Under hypotheses of the lemma, for any $k \in [1, l-2]$, one has the estimate

$$\|du\|^2_{L^2(Z_k)} \leqslant \frac{\gamma_k}{2} \cdot \Big(\|du\|^2_{L^2(Z_{k-1})} + \|du\|^2_{L^2(Z_{k+1})} \Big) \tag{5.3.6}$$

for

$$\gamma_k := \gamma^* + C_1 \cdot \big(e^{-\alpha s k} + e^{-\alpha s (l-k)} \big) \tag{5.3.7}$$

with the parameter $\alpha > 0$ as in Step 0) and some constant C_1 depending only on X, h, J^*, and s.

While proving this estimate we shall denote by C a constant whose particular value is not important and which may not be the same in different formulas. The main condition is that these constants are *uniform*, *i.e.* independent of J, u, and l, and depend only on X, h, J^*, and s.

Estimates (5.3.2) and (5.3.5) together with apriori estimates show that

$$\mathsf{diam}\,(u(Z(k-1, k+2)) \leqslant C \cdot \big(e^{-\alpha k} + e^{-\alpha (l-k)} \big). \tag{5.3.8}$$

Consequently, due to a uniform Hölder $C^{0,s}$-estimate on J, for the oscillation of J on the image $u(Z(k-1, l-k))$ we obtain

$$\mathsf{osc}(J, u(Z(k-1, k+2))) \leqslant C \cdot \big(e^{-\alpha s k} + e^{-\alpha s (l-k)} \big). \tag{5.3.9}$$

This implies that in a neighborhood of each $u(Z(k-1, k+2))$ there exist an integrable structure J_{st} and a flat (*i.e.* Euclidean) metric h_{st} such that

$$\|J - J_{\mathsf{st}}\|_{L^\infty(u(Z_k))} + \|h - h_{\mathsf{st}}\|_{L^\infty(u(Z_k))} \leqslant C \cdot \big(e^{-\alpha s k} + e^{-\alpha s (l-k)} \big). \tag{5.3.10}$$

Using this we obtain estimates

$$\begin{aligned} \|\overline{\partial}_{\mathsf{st}} u\|_{L^2(Z_k)} = \|\overline{\partial}_{\mathsf{st}} u - \overline{\partial}_J u\|_{L^2(Z_k)} \leqslant \|J - J_{\mathsf{st}}\|_{L^\infty(u(Z_k))} \cdot \|du\|_{L^2(Z_k)} \leqslant \\ \leqslant C \cdot \big(e^{-\alpha s k} + e^{-\alpha s (l-k)} \big); \end{aligned} \tag{5.3.11}$$

$$\Big| \|du\|_{L^2(Z_k), h} - \|du\|_{L^2(Z_k), h_{\mathsf{st}}} \Big| \leqslant C \cdot \big(e^{-\alpha s k} + e^{-\alpha s (l-k)} \big) \cdot \|du\|_{L^2(Z_k), h}. \tag{5.3.12}$$

In particular, we can use h_{st} instead of h in our estimates.

Now consider U as a subset on $\mathbb{C}^n$ with the standard J_{st} and h_{st}. Then we can find $u_{\overline{\partial}} \in L^{1,2}(Z(k-1, k+2), \mathbb{C}^n)$ such that $\overline{\partial}_{\mathsf{st}} u_{\overline{\partial}} = \overline{\partial}_{\mathsf{st}} u$ and

$$\|du_{\overline{\partial}}\|_{L^2(Z(k-1,k+2))} \leqslant C \|\overline{\partial}_{\mathsf{st}} u\|_{L^2(Z(k-1,k+2))}. \tag{5.3.13}$$

Set $u_{\mathscr{O}} := u - u_{\overline{\partial}}$, so that $u_{\mathscr{O}}$ is J_{st}-holomorphic. It follows that

$$\|du_{\mathscr{O}}\|^2_{L^2(Z_k)} \leqslant \frac{\gamma^*}{2} \Big(\|du_{\mathscr{O}}\|^2_{L^2(Z_{k-1})} + \|du_{\mathscr{O}}\|^2_{L^2(Z_{k+1})} \Big). \tag{5.3.14}$$

Together with the estimates on $u_{\overline{\partial}}$, (5.3.14) implies (5.3.6).

Step 2). There exist a uniform $k_0 = k_0(X, h, J^*, s)$ and $A^\pm_k$, $k = k_0, \ldots, l - k_0$ with the properties

i) $A^\pm_k$ are "supersolutions" of (5.3.6), i.e.

$$A^\pm_k \geqslant \frac{\gamma_k}{2}(A^\pm_{k-1} + A^\pm_{k+1}) \tag{5.3.15}$$

ii) $A^\pm_k$ have the desired exponential decay

$$A^+_k \leqslant C \cdot e^{-2k}, \qquad\qquad A^-_k \leqslant C \cdot e^{-2(l-k)}. \tag{5.3.16}$$

iii) $\gamma_k < 1$ for $k \in [k_0, l-k_0]$.

Fix $k^* \in \mathbb{Z}$ such that $l-1 \leqslant k^* < l+1$, so that $k^* \approx \frac{l}{2}$. Set

$$A^+_k := \begin{cases} e^{-2k-\frac{1}{k}} & 0 \leqslant k \leqslant k^* \\ e^{-2k-\frac{1}{k^*}+\frac{1}{l-k}-\frac{1}{l-k^*}} & k^* \leqslant k \leqslant l \end{cases} \tag{5.3.17}$$

$$A^-_k := \begin{cases} e^{-2(l-k)-\frac{1}{l-k^*}+\frac{1}{k}-\frac{1}{k^*}} & 0 \leqslant k \leqslant k^* \\ e^{-2(l-k)-\frac{1}{l-k}} & k^* \leqslant k \leqslant l \end{cases} \tag{5.3.18}$$

Making the Taylor expansion in k^{-1} we obtain

$$\frac{2A^\pm_k}{A^\pm_{k-1} + A^\pm_{k+1}} = \begin{cases} \frac{1}{\cosh(2)} + \frac{\sinh(2)}{\cosh^2(2)} \cdot k^{-2} + O(k^{-3}) & \text{for } 0 < k \leqslant k^*; \\ \frac{1}{\cosh(2)} + \frac{\sinh(2)}{\cosh^2(2)} \cdot (l-k)^{-2} + O((l-k)^{-3}) & \text{for } k^* \leqslant k < l; \end{cases}$$

So the existence of the desired $k_0(s)$ follows from the asymptotic behavior $C_1 e^{-\alpha s k} = o(k^{-2})$ for $k \longrightarrow \infty$.

Step 3). There exists a constant $C_2 = C_2(X, h, J^*, s)$ such that

$$\|du\|^2_{L^2(Z_k)} \leqslant C_2 \cdot \Big(A^+_k \cdot \|du\|^2_{L^2(Z(0,2))} + A^-_k \cdot \|du\|^2_{L^2(Z(l-2,l))} \Big) \tag{5.3.19}$$

for any $k \in [k_0, l-k_0]$ with the uniform constant $k_0 = k_0(s)$ chosen as above.

Obviously, (5.3.19) implies the claim of the lemma. Set

$$A^* := C_2 \cdot \Big(A^+_k \cdot \|du\|^2_{L^2(Z(0,2))} + A^-_k \cdot \|du\|^2_{L^2(Z(l-2,l))} \Big)$$

and choose a constant C_2 so that

$$A^*_{k_0} \geqslant \|du\|^2_{L^2(Z_{k_0})} \qquad \text{and} \qquad A^*_{l-k_0} \geqslant \|du\|^2_{L^2(Z_{l-k_0})}.$$

Then by (5.3.6) and (5.3.15)

$$\|du\|^2_{L^2(Z_k)} - A^*_k \leqslant \frac{\gamma_k}{2} \cdot \Big(\|du\|^2_{L^2(Z_{k-1})} - A^*_{k-1} + \|du\|^2_{L^2(Z_{k+1})} - A^*_{k+1} \Big).$$

Find $k_{\mathsf{max}} \in [k_0, l-k_0]$ realizing the maximum of $\|du\|^2_{L^2(Z_k)} - A^*_k$. Then

$$\begin{aligned} \|du\|^2_{L^2(Z_{k_{\mathsf{max}}})} - A^*_{k_{\mathsf{max}}} &\leqslant \tfrac{\gamma_{k_{\mathsf{max}}}}{2} \Big(\|du\|^2_{L^2(Z_{k_{\mathsf{max}}-1})} - A^*_{k-1} + \|du\|^2_{L^2(Z_{k+1})} - A^*_{k_{\mathsf{max}}+1} \Big) \\ &\leqslant \gamma_{k_{\mathsf{max}}} \cdot (\|du\|^2_{L^2(Z_{k_{\mathsf{max}}})} - A^*_{k_{\mathsf{max}}}). \end{aligned} \tag{5.3.20}$$

Since $\gamma_{k_{\mathsf{max}}} < 1$, the last inequality holds only if $\|du\|^2_{L^2(Z_{k_{\mathsf{max}}})} \leqslant A^*_{k_{\mathsf{max}}}$. Thus

$$\|du\|^2_{L^2(Z_k)} \leqslant A^*_k \qquad \text{for any } k \in [k_0, l-k_0]. \tag{5.3.21}$$

This finishes the proof. □

Theorem 5.3.2. *Let J^* be a $C^{0,s}$-smooth almost complex structure on the ball $B \subset \mathbb{R}^{2n}$ with $0 < s < 1$. Then there exists $\varepsilon^* = \varepsilon^*(J^*, s)$ with the following property. For any almost complex structure J on B with $\|J - J^*\|_{C^{0,s}(B)} \leqslant \varepsilon^*$ and any J-holomorphic map $u : Z(0,l) \to B(\frac{1}{2})$ with $l \geqslant 3$ satisfying the condition*

$$\|du\|_{L^2(Z_k)} \leqslant \varepsilon^* \qquad \text{for any } k \in [1,l]$$

there exist a linear complex structure J_{st} in $\mathbb{R}^{2n}$ and vectors $v^+, v^0, v^- \in \mathbb{R}^{2n}$ such that

$$\begin{aligned} \big\| u - \big(e^{-t+J_{\mathsf{st}}\theta} v^+ + v^0 + e^{t-J_{\mathsf{st}}\theta} v^-\big) \big\|^2_{L^{1,2}(Z_k)} &\leqslant \\ \leqslant C^* \cdot \big(k\, e^{-2(1+s)k} \|du\|^2_{L^2(Z(0,2))} &+ (l-k)\, e^{-2(1+s)(l-k)} \|du\|^2_{L^2(Z(l-2,l))} \big) \end{aligned} \tag{5.3.22}$$

for any $k = 1, \ldots, l-1$ with a constant $C^* = C^*(J^*, s) < \infty$ independent of J, l, and u.

Proof. In fact, we prove that for the constant ε^* one can take the ε from *Lemma 5.3.1*. The proof also exploits the same ideas which were used in the proof of that lemma.

Step 1. Let J and $u : Z(0,l) \to B(\frac{1}{2})$ be as in the hypotheses of the theorem. For $k = 1, \ldots, l$, let $x_k \in B(\frac{1}{2})$ be the average value of u on Z_k with respect to the cylinder metric, *i.e.*

$$x_k := \fint_{Z_k} u := \frac{1}{2\pi} \int_{(t,\theta) \in Z_k} u(t,\theta)\, dt\, d\theta.$$

Define the complex structures J_k by $J_k := J(u(k,0))$. We consider every J_k as a *linear* complex structure in $\mathbb{R}^{2n}$, *i.e.* constant in $x \in \mathbb{R}^{2n}$. Further, any $k = 1, \ldots, l$ we define the metric g_k setting $g_k(v,w) := \frac{1}{2}(g_{\mathsf{st}}(v,w) + g_{\mathsf{st}}(J_k v, J_k w))$, where g_{st} denotes the standard Euclidean metric in $\mathbb{R}^{2n}$. Then g_k are linear in the same sense as J_k. In computing various norms related to Z_k or $Z(k-2, k+1)$, we shall use the metric g_k without indicating this in the notation. Observe that all g_k are equivalent since the J_k are uniformly bounded. Further, convergence of J_{k_ν} implies convergence of g_{k_ν}.

For any $k = 1, \ldots, l$ there exist uniquely defined vectors $v_k^+, v_k^0, v_k^- \in \mathbb{R}^{2n}$ such that for the function

$$v_k(t,\theta) := e^{-t+J_k\theta} v_k^+ + v_k^0 + e^{t-J_k\theta} v_k^-$$

the norm $\|u - v_k\|_{L^{1,2}(Z_k)}$ (computed with g_k) attains the minimum.

We claim that under the hypotheses of the theorem there exist a constant $C_1 = C_1(J^*, s)$ and and an integer $k_0 = k_0(J^*, s)$ such that for any integer $k = k_0, \ldots, l - k_0$

$$\begin{aligned} \|u - v_k\|^2_{L^{1,2}(Z_k)} + \|v_{k-1} - v_k\|^2_{L^{1,2}(Z_k)} + \|v_{k+1} - v_k\|^2_{L^{1,2}(Z_k)} &\leqslant \\ \leqslant \gamma_s \cdot \Big(\|u - v_{k-1}\|^2_{L^{1,2}(Z_{k-1})} + \|u - v_{k+1}\|^2_{L^{1,2}(Z_{k+1})} \Big) &+ \\ + C_1 \cdot \big(e^{-2(1+s)k} \|du\|^2_{L^2(Z(0,2))} + e^{-2(1+s)(l-k)} \|du\|^2_{L^2(Z(l-2,l))} \big)& \end{aligned} \tag{5.3.23}$$

with the parameter $\gamma_s := \frac{1}{\cosh(2+2s)}$. Assuming the contrary, there must exist sequences of

- integers $l_\nu \longrightarrow \infty$;
- integers $k_\nu \longrightarrow \infty$ with $l_\nu - k_\nu \longrightarrow \infty$;
- structures J_ν in B with $\|J_\nu - J^*\|_{C^{0,s}(B)} \leqslant \varepsilon^*$;
- J_ν-holomorphic maps $u_\nu : Z(0, l_\nu) \to B(\frac{1}{2})$ with $\sup_{k=0,\ldots,l_\nu} \|du_\nu\|_{L^2(Z_k)} \longrightarrow 0$

with the following property. For the points $x_{\nu,k} := \fint_{Z_k} u_\nu$, the linear complex structure $J_{\nu,k} := J_\nu(x_{\nu,k})$, the corresponding metrics $g_{\nu,k}$, and vectors $v^+_{\nu,k}, v^0_{\nu,k}, v^-_{\nu,k} \in \mathbb{R}^{2n}$ constructed

as above for every $u_\nu|_{Z_k}$ with $k=1,\dots,l_\nu$, at the position $k=k_\nu$ we obtain the inequality in the opposite direction:

$$\|u_\nu - v_{\nu,k_\nu}\|^2_{L^{1,2}(Z_{k_\nu})} + \|v_{k_\nu-1} - v_{k_\nu}\|^2_{L^{1,2}(Z_{k_\nu})} + \|v_{k_\nu+1} - v_{k_\nu}\|^2_{L^{1,2}(Z_{k_\nu})} \geqslant$$
$$\geqslant \gamma_s \cdot \Big(\|u_\nu - v_{\nu,k_\nu-1}\|^2_{L^{1,2}(Z_{k_\nu-1})} + \|u_\nu - v_{\nu,k_\nu+1}\|^2_{L^{1,2}(Z_{k_\nu+1})} \Big) +$$
$$+ \nu \cdot \big(e^{-2(1+s)k_\nu} \|du_\nu\|^2_{L^2(Z(0,2))} + e^{-2(1+s)(l-k_\nu)} \|du_\nu\|^2_{L^2(Z(l_\nu-2,l_\nu))} \big). \qquad (5.3.24)$$

Let us estimate the behavior of $u_\nu - v_{\nu,k}$ in Z_k for $k \approx k_\nu$. Set

$$A_{\nu,k} := e^{-k}\|du_\nu\|_{L^2(Z(0,2))} + e^{-(l-k)}\|du_\nu\|_{L^2(Z(l_\nu-2,l_\nu))}.$$

Then by *Lemma 5.3.1* we have $\|du_\nu\|_{L^2(Z_k)} \leqslant C \cdot A_{\nu,k}$. This yields a similar estimate on the diameter: $\mathsf{diam}\,(u_\nu(Z_k)) \leqslant C \cdot A_{\nu,k}$, possibly with a new constant C. Further, for a linear complex structure J' with the corresponding operator $\overline{\partial}' := \overline{\partial}_{J'}$ we obtain the pointwise estimate

$$\left|\overline{\partial}' u_\nu\right| = \left|\overline{\partial}' u_\nu - \overline{\partial}_{J_\nu} u_\nu\right| = |(\partial_x u_\nu - J' \cdot \partial_y u_\nu) - (\partial_x u_\nu - J_\nu(u_\nu) \cdot \partial_y u_\nu)|$$
$$\leqslant |J' - J_\nu \circ u_\nu| \cdot |du_\nu|.$$

For $J_{\nu,k}$ this yields the estimate

$$\|\overline{\partial}_{J_{\nu,k}} u_\nu\|_{L^2(Z(k-2,k+1))} \leqslant C\big(\mathsf{diam}\,(u_\nu(Z(k-2,k+1)))\big)^s \cdot \|du_\nu\|_{L^2(Z(k-2,k+1))} \leqslant C' \cdot A^{1+s}_{\nu,k}. \qquad (5.3.25)$$

By construction, $J_{\nu,k}$ are uniformly bounded. This implies that we can represent $u_\nu|_{Z_k}$ in the form $u_\nu|_{Z_k} = w_{\nu,k} + f_{\nu,k}$, where $w_{\nu,k}$ is $J_{\nu,k}$-holomorphic and $f_{\nu,k}$ is estimated as

$$\|f_{\nu,k}\|_{L^{1,2}(Z_k)} \leqslant C \cdot A^{1+s}_{\nu,k}.$$

Define the positive η_ν by the relation

$$\eta_\nu^2 = \|u_\nu - v_{\nu,k_\nu}\|^2_{L^{1,2}(Z_{k_\nu})} + \|v_{k_\nu-1} - v_{k_\nu}\|^2_{L^{1,2}(Z_{k_\nu})} + \|v_{k_\nu+1} - v_{k_\nu}\|^2_{L^{1,2}(Z_{k_\nu})}$$

and set

$$\tilde u_\nu(t,\theta) := \frac{1}{\eta_\nu} u_\nu(t+k_\nu,\theta), \qquad \tilde w_{\nu,k}(t,\theta) := \frac{1}{\eta_\nu} w_{\nu,k+k_\nu}(t+k_\nu,\theta),$$

$$\tilde f_{\nu,k}(t,\theta) := \frac{1}{\eta_\nu} f_{\nu,k+k_\nu}(t+k_\nu,\theta), \qquad \tilde J_{\nu,k} := J_{\nu,k+k_\nu}, \qquad \tilde v^\iota_{\nu,k} := \frac{1}{\eta_\nu} v^\iota_{\nu,k+k_\nu},\ \iota = +,0,-,$$

In other words, we shift all the picture from Z_{k_ν} to Z_0 and rescale the maps u_ν, the vectors $v^\iota_{\nu,k}$, $\iota=+,0,-$, and so on in a way as to make the left hand side of (5.3.24) equal to 1.

It follows from (5.3.24) that $A^{1+s}_{\nu,k+k_\nu} \leqslant C\nu^{-1/2}\eta_\nu = o(\eta_\nu)$ for any fixed k. Consequently, $\|\tilde f_{\nu,k}\|_{L^{1,2}(Z_k)} \longrightarrow 0$ for any fixed k and $\nu \longrightarrow \infty$. This implies that the norms $\|\tilde w_{\nu,k} - \tilde v_{\nu,k}\|_{L^{1,2}(Z_k)}$ remain uniformly bounded in ν for any fixed k.

Represent every $\tilde w_{\nu,k}$ as the Laurent series

$$\tilde w_{\nu,k}(t,\theta) = \textstyle\sum_{m=-\infty}^{+\infty} e^{m(-t+J_{\nu,k}\theta)} w^m_{\nu,k} \qquad (5.3.26)$$

and denote by $\tilde w'_{\nu,k}$ the sum of terms with $m=0,\pm1$, *i.e.*

$$\tilde w'_{\nu,k}(t,\theta) := e^{t-J_{\nu,k}\theta} w^{-1}_{\nu,k} + w^0_{\nu,k} + e^{-t+J_{\nu,k}\theta} w^1_{\nu,k}.$$

It follows from the construction of $\tilde v_{\nu,k}$ that

$$\|\tilde w'_{\nu,k} - \tilde v_{\nu,k}\|_{L^{1,2}(Z_k)} = O(\|\tilde f_{\nu,k}\|_{L^{1,2}(Z_k)}) \longrightarrow 0 \qquad (5.3.27)$$

for any fixed k. Indeed, $\tilde v_{\nu,k}$, considered as a function in θ, is a linear combination of a constant and the trigonometric functions $\mathsf{cos}\theta$ and $\mathsf{sin}\theta$. So it is orthogonal to the remaining terms $e^{m(-t+J_{\nu,k}\theta)}w^m_{\nu,k}$, $|m|\geqslant 2$. Thus $\tilde w'_{\nu,k}$ is the best approximation of $\tilde w_{\nu,k}$ by such linear combinations, whereas the difference $\tilde w'_{\nu,k}-\tilde v_{\nu,k}$ appears as the best approximation of $\tilde f_{\nu,k}$.

Since $\tilde J_{\nu,0}$ is bounded uniformly in ν, there exists a subsequence, still indexed by ν, which converges to a linear complex structure $\tilde J$. It follows from the definition of $\tilde J_{\nu,k}$ and the estimate on the diameter of $u_\nu(Z_k)$ for $k\approx k_\nu$ that for any fixed k the structures $\tilde J_{\nu,k}$ also converge to $\tilde J$.

Now we show that, after going to a subsequence, $\tilde u_{\nu,0}-\tilde v_{\nu,0}$ converges weakly in the $L^{1,2}(Z(-2,1))$-topology to a $\tilde J$-holomorphic function, and that this convergence is strong in the $L^{1,2}(Z_0)$-topology. The inequality (5.3.24) together with the choice of η_ν and the construction of $\tilde u_{\nu,k}$ gives boundedness of the norms $\|\tilde u_{\nu,0}-\tilde v_{\nu,0}\|_{L^{1,2}(Z(-2,1))}$ uniform in ν. So the weak convergence follows. From (5.3.25) and $A^{1+s}_{\nu,k}=o(\eta_\nu)$ we obtain the vanishing $\|\overline\partial_{\tilde J_{\nu,0}}\tilde u_{\nu,0}\|_{L^{1,2}(Z(-2,1))}\longrightarrow 0$. The estimate (5.3.27) and $\|\tilde f_{\nu,k}\|_{L^{1,2}(Z_k)}\longrightarrow 0$ yield $\|\overline\partial_{\tilde J_{\nu,0}}\tilde v_{\nu,0}\|_{L^{1,2}(Z(-2,1))}\longrightarrow 0$. Now the desired strong $L^{1,2}(Z_0)$-convergence follows from elliptic regularity of $\tilde J_{\nu,0}$. In the same way for $k=\pm1$ we obtain the weak $L^{1,2}$-convergence of $\tilde u_{\nu,k}-\tilde v_{\nu,k}$ in Z_k.

For $k=0,\pm1$, let $\tilde u_k:=\mathsf{lim}\,\tilde u_{\nu,k}-\tilde v_{\nu,k}$ be the limit functions obtained above. Since $\|\tilde f_{\nu,k}\|_{L^{1,2}(Z_k)}\longrightarrow 0$, these are $\tilde J$-holomorphic functions in Z_k, $\tilde J=\mathsf{lim}\,\tilde J_{\nu,k}$.

Observe that the functions $\tilde v_{\nu,\pm1}-\tilde v_{\nu,0}$ are linear combinations of constants and the function $e^{\pm t}\mathsf{cos}\theta$, $e^{\pm t}\mathsf{sin}\theta$ which are uniformly bounded in the $L^{1,2}(Z_0)$-norm. Consequently, after taking a subsequence, we also obtain the strong $L^{1,2}$-convergence in Z_0. This implies the strong $L^{1,2}$-convergence in $Z(-2,1)$. Finally, from (5.3.27) we conclude that the Laurent series for each $\tilde u_k$ does not contain terms of degree $m=0,\pm1$, *i.e.* a constant term and a multiple of $e^{\pm(-t+\tilde J\theta)}$. This, in turn, implies that, first, the $\tilde u_k(t,\theta)$ are restrictions to Z_k of the same $\tilde J$-holomorphic function $\tilde u$, and second, $\mathsf{lim}\,\tilde v_{\nu,-1}-\tilde v_{\nu,0}=\mathsf{lim}\,\tilde v_{\nu,1}-\tilde v_{\nu,0}=0$.

Substituting into (5.3.24), we see that $\tilde u$ satisfies the inequality

$$\|\tilde u\|^2_{L^{1,2}(Z_0)}\geqslant\gamma_s\cdot\Big(\|\tilde u\|^2_{L^{1,2}(Z_{-1})}+\|\tilde u\|^2_{L^{1,2}(Z_1)}\Big). \tag{5.3.28}$$

On the other hand, the absence of the terms of degree $m=0,\pm1$ in the Laurent decomposition of type (5.3.26) for $\tilde u$ implies the inequality

$$\|\tilde u\|^2_{L^{1,2}(Z_0)}\leqslant\gamma_2\cdot\Big(\|\tilde u\|^2_{L^{1,2}(Z_{-1})}+\|\tilde u\|^2_{L^{1,2}(Z_1)}\Big). \tag{5.3.29}$$

with $\gamma_2:=\frac{1}{\mathsf{cosh}(4)}$. This inequality is easily obtained for mutually orthogonal terms $\tilde u^m e^{m(-t+\tilde J\theta)}$. However, since $s<1$, $\gamma_2<\gamma_s=\frac{1}{\mathsf{cosh}(2+2s)}$, which is a contradiction.

This implies the validity of (5.3.23) for all $k=k_0,\dots,l-k_0$ with k_0 independent of J, l, and u.

Step 2. We now turn back to the proof of the theorem. To show that (5.3.23) implies (5.3.22), we set for $k=0,\dots,l$

$$A'_k:=k\cdot\frac{\mathsf{cosh}(2+2s)}{\mathsf{sinh}(2+2s)}\cdot C_1\cdot e^{-2(1+s)k}\|du\|^2_{L^2(Z(0,2))}+ \\ +(l-k)\cdot\frac{\mathsf{cosh}(2+2s)}{\mathsf{sinh}(2+2s)}\cdot C_1\cdot e^{-2(1+s)(l-k)}\|du\|^2_{L^2(Z(l-2,l))}. \tag{5.3.30}$$

Then A'_k satisfies the equality

$$A'_k = \frac{\gamma_s}{2}\cdot(A'_{k-1}+A'_{k+1}) + C_1\cdot\big(e^{-2(1+s)k}\|du\|^2_{L^2(Z(0,2))} + e^{-2(1+s)(l-k)}\|du\|^2_{L^2(Z(l-2,l))}\big).$$

Consequently,

$$\begin{aligned}\|u-v_k\|^2_{L^{1,2}(Z_k)} - A'_k + \|v_{k-1}-v_k\|^2_{L^{1,2}(Z_k)} + \|v_{k+1}-v_k\|^2_{L^{1,2}(Z_k)} \leqslant\\ \leqslant \gamma_s\cdot\Big(\|u-v_{k-1}\|^2_{L^{1,2}(Z_{k-1})} - A'_{k-1} + \|u-v_{k+1}\|^2_{L^{1,2}(Z_{k+1})} - A'_{k+1}\Big).\end{aligned} \tag{5.3.31}$$

As in the proof of *Lemma 5.3.1*, (5.3.31) implies the estimate

$$\begin{aligned}\|u-v_k\|^2_{L^{1,2}(Z_k)} - A'_k \leqslant\\ \leqslant C_2\cdot\big(e^{-2(1+s)k}\|du\|^2_{L^2(Z(0,2))} + e^{-2(1+s)(l-k)}\|du\|^2_{L^2(Z(l-2,l))}\big)\end{aligned} \tag{5.3.32}$$

for all $k=k_0,\dots,l-k_0$ with k_0 independent of J, l, and u. Substitution the definition of A'_k yields

$$\begin{aligned}\|u-v_k\|^2_{L^{1,2}(Z_k)} + \|v_{k-1}-v_k\|^2_{L^{1,2}(Z_k)} + \|v_{k+1}-v_k\|^2_{L^{1,2}(Z_k)} \leqslant\\ \leqslant C_3\cdot\big(k\cdot e^{-2(1+s)k}\|du\|^2_{L^2(Z(0,2))} + (l-k)\cdot e^{-2(1+s)(l-k)}\|du\|^2_{L^2(Z(l-2,l))}\big)\end{aligned} \tag{5.3.33}$$

Step 3. For concrete J, l, and u as in the hypotheses of the theorem, find k^* for which the right hand side of (5.3.33) takes its minimum. Set $J_{\mathsf{st}} := J_{k^*} = J(x_{k^*})$, and $v^\iota := v^\iota_{k^*}$, $\iota=-,0,+$, and $v(t,\theta) := v_{k^*}(t,\theta)$. In view of (5.3.33), for the proof of the theorem it is sufficient to estimate $\|v_k - v\|_{L^{1,2}(Z_k)}$.

We do this by descending recursion starting from $k=k^*$. Assume that we have shown that

$$\|v_k - v\|_{L^{1,2}(Z_k)} \leqslant C_4\, k^{1/2}\, e^{-(1+s)k}\,\|du\|_{L^2(Z(0,2))} \tag{5.3.34}$$

for all $k=k_1+1,\dots,k^*$ with the constant C_4 to be chosen below. By our choice of k^*, for $k=1,\dots,k_1$ we obtain from (5.3.33)

$$\|v_k - v_{k+1}\|_{L^{1,2}(Z_k)} \leqslant 2\,C_3\, k^{1/2}\, e^{-(1+s)k}\,\|du\|_{L^2(Z(0,2))}.$$

Observe that for any function $w(t,\theta)$ of the form

$$w(t,\theta) = w^0 + (e^t w_c^+ + e^{-t} w_c^-)\cos(\theta) + (e^t w_s^+ + e^{-t} w_s^-)\sin(\theta)$$

with constant vectors $w^0, w_c^\pm, w_s^\pm \in \mathbb{R}^{2n}$—so are all our differences $v_k - v_{k'}$—we have the estimate

$$\|w\|_{L^{1,2}(Z_k)} \leqslant e\cdot\|w\|_{L^{1,2}(Z_{k+1})}.$$

Applying this, we obtain

$$\begin{aligned}\|v_{k_1} - v\|_{L^{1,2}(Z_{k_1})} &\leqslant \|v_{k_1} - v_{k_1+1}\|_{L^{1,2}(Z_{k_1})} + \|v_{k_1+1} - v\|_{L^{1,2}(Z_{k_1})}\\ &\leqslant 2\,C_3\, k_1^{1/2}\, e^{-(1+s)k_1}\,\|du\|_{L^2(Z(0,2))} + e\cdot\|v_{k_1+1} - v\|_{L^{1,2}(Z_{k_1+1})}\\ &\leqslant 2\,C_3\, k_1^{1/2}\, e^{-(1+s)k_1}\,\|du\|_{L^2(Z(0,2))} + C_4\,(k_1+1)^{1/2}\, e^{1-(1+s)(k_1+1)}\,\|du\|_{L^2(Z(0,2))}\\ &= 2\,C_3\, k_1^{1/2}\, e^{-(1+s)k_1}\,\|du\|_{L^2(Z(0,2))} + e^{-s}\,C_4\,(k_1+1)^{1/2}\, e^{-(1+s)k_1}\,\|du\|_{L^2(Z(0,2))}.\end{aligned}$$

Assume additionally that $e^{-s/2}(k_1+1)^{1/2} \leqslant k_1^{1/2}$. Then setting $C_4 := \frac{2C_3}{1-e^{-s/2}}$ we can conclude that (5.3.34) also holds for $k=k_1$. Since the condition $e^{-s/2}(k_1+1)^{1/2} \leqslant \cdot k_1^{1/2}$ is equivalent to $k_1 \geqslant \frac{1}{e^s-1}$, our recursive construction implies (5.3.34) for all $k\in\big[\frac{1}{e^s-1}, k^*\big]$. For the remaining $k\in\big[1,\frac{1}{e^s-1}\big]$ the estimate (5.3.34) follows from *Lemma 5.3.1*.

Making a similar recursive construction for $k = k^*, \dots, l$ we obtain the estimate

$$\|v_k - v\|_{L^{1,2}(Z_k)} \leqslant C_4\,(l-k)^{1/2}\, e^{-(1+s)(l-k)}\, \|du\|_{L^2(Z(l-2,l))} \tag{5.3.35}$$

for all $k = k^*, \dots, l-1$ with the the same constant C_4. Now (5.3.34), (5.3.35), and (5.3.33) imply the desired estimate (5.3.22). □

5.4. **Deformation of a node and gluing.** The cycle topology on $\overline{\mathscr{M}}$, introduced in *Paragraph 5.2*, has the nice property that $\mathsf{pr}_{\mathscr{J}} : \overline{\mathscr{M}} \to \mathscr{J}$ is continuous and *proper*. The last property is follows from the Gromov compactness for closed curves. However, it is desirable to have a better understanding of the topological structure of $\overline{\mathscr{M}}$. Recall that in *Paragraph 5.2* we obtained a natural stratification of $\overline{\mathscr{M}}$ in which the strata are distinguished by a topological type of curves. Moreover, every stratum $\mathscr{M}_{\boldsymbol{\tau}}$ has a natural structure of a C^ℓ-smooth Banach manifold such that the restricted projection $\mathsf{pr}_{\mathscr{J}} : \mathscr{M}_{\boldsymbol{\tau}} \to \mathscr{J}$ is Fredholm. So to understand of the topology of $\overline{\mathscr{M}}$ means to describe how different strata are attached to each other. The most important problem is to describe deformations of the standard node. Let us formulate the question as follows:

Gluing problem. *Let $J_0 \in \mathscr{J}$ be an almost complex structure and $u_0 : \mathscr{A}_0 \to X$ a J_0-holomorphic map. Describe possible J-holomorphic maps $u : Z(0,l) \to X$ with $J \in \mathscr{J}$ sufficiently close to J_0 and $l \gg 0$ which are sufficiently close to u_0 with respect to the Gromov topology.* In other words, we try to reverse the bubbling and construct a single map u of a long cylinder $Z(0,l)$ by gluing together the components $u_0', u_0'' : \Delta \to X$ of the map u_0.

Moreover, one would like to have a smooth structure on the set of such deformation, so that the transversality techniques could be applied. This means that one seeks a family of deformations of a given $u_0 : \mathscr{A}_0 \to (X, J_0)$ depending smoothly on the parameter.

As the main result of this paragraph we give a satisfactory solution to the *Gluing problem*. Let us start with introducing some notation.

Definition 5.4.1. For a fixed sufficiently small $\varepsilon > 0$, let

$$\mathscr{A} := \{\, (z^+, z^-) \in \Delta^2 \;:\; |z^+| \cdot |z^-| < \varepsilon \,\} \tag{5.4.1}$$

with the projection

$$\mathsf{pr}_{\mathscr{A}} : \mathscr{A} \to \Delta(\varepsilon), \quad \mathsf{pr}_{\mathscr{A}}(z^+, z^+) = \lambda(z^+, z^+) := z^+ \cdot z^-. \tag{5.4.2}$$

Further, for $\lambda \in \Delta(\varepsilon)$ define the analytic sets

$$\mathscr{A}_\lambda := \{\, (z^+, z^-) \in \Delta^2 \;:\; z^+ \cdot z^- = \lambda \,\} = \mathsf{pr}_{\mathscr{A}}^{-1}(\lambda). \tag{5.4.3}$$

For $\lambda = 0$ this is the standard node and for $\lambda \neq 0$ a cylinder of conformal radius $R = \log \frac{1}{|\lambda|}$. Define

$$\mathscr{A}_\lambda^\pm := \{(z^+, z^-) \in \mathscr{A}_\lambda : |\lambda| \leqslant z^\pm < 1\}.$$

Then $\mathscr{A}_\lambda^\pm$ are subannuli for $\lambda \neq 0$ and $\mathscr{A}_0^\pm$ are discs $\Delta^\pm$, the irreducible components of $\mathscr{A}_0$. In any case, $\mathscr{A}_\lambda = \mathscr{A}_\lambda^+ \cup \mathscr{A}_\lambda^-$.

To describe a Hermitian metric on a complex manifold X, it is sufficient to indicate only the corresponding Kähler form ω. In this case we shall say that ω *induces* a metric on X or even that ω *is* a metric on X. The author begs the reader's pardon for such informality in the terminology. The advantage of such notation is that the restriction of a metric on a complex submanifold is given by the restriction of the corresponding Kähler form. In this notation, the standard metric on the disc Δ with the coordinate z is given by the form $\frac{\mathsf{i}}{2} dz \wedge d\bar{z}$.

We equip $\mathscr{A}_\lambda$ with the Riemannian metric induced from Δ^2. This gives the standard metric $\frac{\mathrm{i}}{2}dz^\pm \wedge d\bar z^\pm$ on each component $\Delta^\pm$ of $\mathscr{A}_0$ and the *hyperbola metric*

$$\frac{\mathrm{i}}{2}\Big(1+\frac{|\lambda|^2}{|z^+|^4}\Big)dz^+\wedge d\bar z^+ = \frac{\mathrm{i}}{2}\Big(1+\frac{|\lambda|^2}{|z^-|^4}\Big)dz^-\wedge d\bar z^- \tag{5.4.4}$$

on $\mathscr{A}_\lambda$ with $\lambda \neq 0$.

Set

$$\check{\mathscr{A}} := \{\,(z^+,z^-)\in\mathscr{A} \;:\; z^+\cdot z^-\neq 0\,\} = \sqcup_{\lambda\in\check\Delta(\varepsilon)}\mathscr{A}_\lambda$$

and

$$V^\pm := \{1-\varepsilon<|z^\pm|<1\}, \qquad V := V^+\sqcup V^-.$$

For a given λ we have the canonical imbedding $V \to \mathscr{A}_\lambda$, defined by the coordinate functions $z^\pm$ on $\mathscr{A}_\lambda$ and on $V^\pm$. This imbedding defines the restriction map

$$u\in L^{1,p}(\mathscr{A}_\lambda,X)\mapsto u|_V\in L^{1,p}(V,X) \qquad u\mapsto (u(z^+)|_{V^+},\, u(z^-)|_{V^-}).$$

Definition 5.4.2. For a nodal curve C with smooth boundary $\partial C=\sqcup_i\gamma_i$, $\gamma_i\cong S^1$, let $\mathscr{P}(C)$ be the set of *stable pseudoholomorphic maps between C and X*,

$$\mathscr{P}(C) := \{\,(u,J)\in L^{1,p}(C,X)\times\mathscr{J} \;:\; \bar\partial_J u=0,\ \ u \text{ is stable}\,\}. \tag{5.4.5}$$

Equip $\mathscr{P}(C)$ with the topology induced from $L^{1,p}(C,X)\times\mathscr{J}$. In particular, $\mathscr{P}(V)$ consists of triples (u^+,u^-,J), where $u^\pm : V^\pm\to X$ is $L^{1,p}$-smooth J-holomorphic map. Denote by $\mathscr{P}^*(C)$ the subset of $(u,J)\in\mathscr{P}(C)$ for which u is *non-multiple on the union of compact components of C*. Further, we define

$$\mathscr{P}(\mathscr{A}) := \sqcup_{\lambda\in\Delta(\varepsilon)}\mathscr{P}(\mathscr{A}_\lambda), \qquad \mathscr{P}(\check{\mathscr{A}}) := \sqcup_{\lambda\in\check\Delta(\varepsilon)}\mathscr{P}(\mathscr{A}_\lambda) \tag{5.4.6}$$

and equip this spaces with the topology induced by the Gromov convergence in the interior of $\mathscr{A}_\lambda$ and $L^{1,p}$-convergence near boundary. This means that (u_n,J_n,λ_n) converges to $(u_\infty,J_\infty,\lambda_\infty)$ if (J_n,λ_n) converges to $(J_\infty,\lambda_\infty)$ in $\mathscr{J}\times\Delta(\varepsilon)$, the restrictions $u_n|_V$ converges to $u_\infty|_V$ with respect to $L^{1,p}$-norm, and u_n converges to u_∞ in the sense of *Definition 5.2.7*. Elements of $\mathscr{P}(\mathscr{A})$ will be denoted by (u,J,λ). As usual, $\mathsf{pr}_{\mathscr{J}}$ stands for the natural projections from $\mathscr{P}(C)$ or $\mathscr{P}(\mathscr{A})$ to $\mathscr{J}$.

Theorem 5.4.1. *The natural map $\mathsf{pr}_V:\mathscr{P}(\mathscr{A})\to\mathscr{P}(V)\times\Delta(\varepsilon)$, defined by*

$$\mathsf{pr}_V(u,J,\lambda) := (u(z^+)|_{V^+}, u(z^-)|_{V^-}, J;\lambda) \tag{5.4.7}$$

is an imbedding of a topological Banach submanifold.

Moreover, for every $(u_0,J_0)\in\mathscr{P}(\mathscr{A}_0)$ there exists a neighborhood $\mathscr{U}\subset\mathscr{P}(\mathscr{A}_0)$ of (u_0,J_0), an $\varepsilon'>0$, and a map $\Phi:\mathscr{U}\times\Delta(\varepsilon')\to\mathscr{P}(\mathscr{A})$ such that

- *Φ is a homeomorphism onto the image;*
- *for every $\lambda\in\Delta(\varepsilon')$ the restricted map $\Phi_\lambda:=\Phi|_{\mathscr{U}\times\{\lambda\}}:\mathscr{U}\to\mathscr{P}(V)\times\Delta(\varepsilon)$takes values in $\mathscr{P}(\mathscr{A}_\lambda)\subset\mathscr{P}(\mathscr{A})$ and is a C^1-diffeomorphism;*
- *the family of maps $\Psi_\lambda:=\mathsf{pr}_V\circ\Phi_\lambda:\mathscr{U}\to\mathscr{P}(V)$ depends continuously on $\lambda\in\Delta(\varepsilon')$ with respect to the C^1-topology.*

In other words, the theorem statess that $\mathscr{P}(\mathscr{A})=\sqcup\mathscr{P}(\mathscr{A}_\lambda)$ is a continuous family of C^1-submanifolds. Before proving this result, we state and prove a corollary which provides a technique which allows one to smooth nodal points on pseudoholomorphic curves.

Theorem 5.4.2. *Let C^* be a closed connected nodal curve parameterized by a real surface S, $J^* \in \mathscr{J}$, $u^* : C^* \to X$ a J^*-holomorphic map, and $(C^*, u^*, J^*) \in \overline{\mathscr{M}}^{Gr} = \overline{\mathscr{M}}^{Gr}(S, X, [C^*])$ the corresponding element of the Gromov compactification of the total moduli space. Assume that the map $u^* : C^* \to X$ is non-multiple.*

Then there exist $(C', u', J') \in \mathscr{M}(S, X, [C^*])$ arbitrarily close to (C^*, u^*, J^*) with respect to the Gromov topology such that C' is a smooth curve.

The notation used in this theorem was introduced in *Definitions 5.1.8* and *5.2.2*. Note that the condition of non-multiplicity of $u^* : C^* \to X$ is equivalent to the absence of ghost and multiple components.

Proof. Let $\{z_1^*, \ldots z_k^*\}$ be the set of nodal points of C^*. For every z_i^* fix a neighborhood V_i isomorphic to the standard node. We may also assume that the sets V_i are pairwise disjoint. Let u_i^* denote the restriction of u^* to V_i. Applying *Theorem 5.4.1* we can perturb $V_i \cong \mathscr{A}_0$ to an annulus $V_i' = \mathscr{A}_{\lambda_i}$ and $u_i^* : V_i \to X$ to a J^*-holomorphic map $u_i' : V_i' \to X$. If these perturbations (V_i', u_i') are made small enough, then we can adjust the structure J^* and the map u^* on the remaining part of the curve C^* in a way yielding the desired $(C', u', J') \in \mathscr{M}(S, X, [C^*])$. $\square$

Modifying the proof of *Theorem 5.4.2* one can also obtain

Proposition 5.4.3. *Let C^* be a closed connected nodal curve parameterized by a real surface S, $J^* \in \mathscr{J}$, $u^* : C^* \to X$ a J^*-holomorphic map, and $(C^*, u^*, J^*) \in \overline{\mathscr{M}}^{Gr} = \overline{\mathscr{M}}^{Gr}(S, X, [C^*])$ the corresponding element of the Gromov compactification of the total moduli space. Assume that the map $u^* : C^* \to X$ is non-multiple.*

Then in a neighborhood of (C^*, u^*, J^*) the space $\overline{\mathscr{M}}^{Gr}$ is a topological Banach manifold and the natural projection $\mathsf{pr}^{Gr} : \overline{\mathscr{M}}^{Gr} \to \overline{\mathscr{M}}$ a homeomorphism.

Since the result of *Proposition 5.4.3* is not needed for the purposes of this paper, we leave it without a proof.

The proof of *Theorem 5.4.1* is divided in the subsequent lemmas. The first two are simple but useful technical results.

Lemma 5.4.4. *Let C be a nodal curve without closed compact components, and E a holomorphic vector bundle over C. Then any operator $D : L^{1,p}(C, E) \to L^p_{(0,1)}(C, E)$ of the form $D = \overline{\partial}_E + R$ with $R \in L^p$ is surjective and its kernel $\mathsf{H}^0_D(C, E)$ admits a closed complement.*

Proof. Imbed C into a compact nodal curve $\widetilde{C}$ and extend E to a holomorphic vector bundle $\widetilde{E}$ over $\widetilde{C}$. Without loss of generality we may assume that the Chern numbers $\langle c_1(\widetilde{E}), \widetilde{C}_i \rangle$ are sufficiently large for each component $\widetilde{C}_i$ of $\widetilde{C}$. Now extend $R \in L^p(C, \mathsf{Hom}_{\mathbb{R}}(E, E \otimes \Lambda^{(0,1)}))$ to $\widetilde{R} \in L^p(\widetilde{C}, \mathsf{Hom}_{\mathbb{R}}(\widetilde{E}, \widetilde{E} \otimes \Lambda^{(0,1)}))$ and set $\widetilde{D} := \overline{\partial}_{\widetilde{E}} + \widetilde{R}$. Adjusting $\widetilde{R}$ on the complement $\widetilde{C} \backslash C$ we may assume that $\widetilde{D} : L^{1,p}(\widetilde{C}, \widetilde{E}) \to L^p_{(0,1)}(\widetilde{C}, \widetilde{E})$ is surjective. The sufficient condition for existence of such an adjustment is provided by the condition on the Chern numbers of $\widetilde{E}$. Since any $\eta \in L^p_{(0,1)}(C, E)$ extends to $\widetilde{\eta} \in L^p_{(0,1)}(\widetilde{C}, \widetilde{E})$, the surjectivity of $\widetilde{D}$ implies the surjectivity of D.

The existence of a closed complement to the kernel of D is shown in [Iv-Sh-2] in the case when $D = \overline{\partial}$. This proof applies also in our case with only minor adjustments. $\square$

Remark. The existence of a closed complement to the kernel of D allows us to apply the implicit function theorem.

Lemma 5.4.5. *i) The space $L^{1,p}(C,X)$ is a smooth Banach manifold with tangent space*

$$T_u L^{1,p}(C,X) = L^{1,p}(C,E_u). \tag{5.4.8}$$

ii) The space $\mathscr{P}^*(C)$ is a C^ℓ-smooth submanifold of $L^{1,p}(C,X)\times\mathscr{J}$ with tangent space

$$T_{(u,J)}\mathscr{P}^*(C) = \{(v,\dot J)\in L^{1,p}(C,E_u)\times T_J\mathscr{J} \,:\, D_{u,J}v+\dot J\circ du\circ J_C=0\}. \tag{5.4.9}$$

iii) $\mathsf{pr}_V : \mathscr{P}(\mathscr{A}_0)\to\mathscr{P}(V)$ and $\mathsf{pr}_V : \mathscr{P}(\check{\mathscr{A}})\to\mathscr{P}(V)\times\check\Delta(\varepsilon)$ are C^ℓ-smooth imbeddings on Banach submanifolds.

The definitions of the spaces which are involved here are given in *Definition 5.1.5* and *Definition 5.4.2*.

Proof. Let $C=\cup C_i$ be the decomposition of C into components and $\{(z'_a,z''_a)\}$ the set of pairs of points on the normalization $\hat C$ corresponding to the nodal points.

i) The space $L^{1,p}(C,X)$ is a subset of a smooth Banach manifold $\prod_i L^{1,p}(C_i,X)$ defined by equations $u(z'_a)=u(z''_a)$. One checks the transversality condition and computes the tangent space.

ii) One can use the same arguments as in part *i)*.

iii) First we note that *Lemma 1.2.5 ii)* implies the following unique continuation property: Any J-holomorphic map u, defined on an open set U of a nodal curve C admits at most one J-holomorphic extension to an irreducible component C' of C provided $C'\cap U\neq\varnothing$. This shows that the restriction maps $F_0 : \mathscr{P}(\mathscr{A}_0)\to\mathscr{P}(V)$ and $\check F : \mathscr{P}(\check{\mathscr{A}})\to\mathscr{P}(V)\times\check\Delta(\varepsilon)$ are set-theoretically injective. Note that we have introduced new notation, F_0 and $\check F$, for the restrictions of the map pr_V to the corresponding definition domains.

Further, $F_0 : \mathscr{P}(\mathscr{A}_0)\to\mathscr{P}(V)$ is obviously C^ℓ-smooth and the differential $dF_0 : T_u\mathscr{P}(\mathscr{A}_0)\to T_u\mathscr{P}(V)$ is simply the restriction map $dF_0 : (v,\dot J)\mapsto(v|_V,\dot J)$. *Lemma 5.4.4* shows that the restriction map

$$\{\,v\in L^{1,p}(\mathscr{A}_0,E_u)\,:\,D_{u,J}v=0\,\}\mapsto\{\,v\in L^{1,p}(V,E_u)\,:\,D_{u,J}v=0\,\} \tag{5.4.10}$$

is a closed imbedding and splits, *i.e.* admits a closed complement.

The claim about $\check F : \mathscr{P}(\check{\mathscr{A}})\to\mathscr{P}(V)\times\Delta(\varepsilon)$ is proven in a similar way. Details are left to the reader. □

A crucial point in the proof of *Theorem 5.4.1* is to find an apriori estimate for the operator $D_{u,J,\lambda}$ which is uniform as $\lambda\longrightarrow 0$. Because of local nature of the estimate it is sufficient to work with the ball $B\subset\mathbb{R}^{2n}\cong\mathbb{C}^n$ equipped with the standard complex structure J_{st}. We start with introducing a chart for the space $L^{1,p}(\mathscr{A},\mathbb{C}^n):=\sqcup_{\lambda\in\Delta(\varepsilon)}L^{1,p}(\mathscr{A}_\lambda,\mathbb{C}^n)$.

Definition 5.4.3. For a nodal complex curve C and a complex manifold X let $\mathscr{H}(C,X)$ denote the space of holomorphic maps $f : C\to X$ which are $L^{1,p}$-smooth up to the boundary ∂C. In the case $X=\mathbb{C}$ we abbreviate the notation to $\mathscr{H}(C)$.

Lemma 5.4.6. [3] *There exist families of homomorphisms $T_\lambda : L^p_{(0,1)}(\mathscr{A}_\lambda,\mathbb{C})\to L^{1,p}(\mathscr{A}_\lambda,\mathbb{C})$ and isomorphisms $\mathsf{L}_\lambda : \mathscr{H}(\mathscr{A}_0)\to\mathscr{H}(\mathscr{A}_\lambda)$ and $Q_\lambda : L^p_{(0,1)}(\mathscr{A}_0,\mathbb{C})\to L^p_{(0,1)}(\mathscr{A}_\lambda,\mathbb{C})$ with the following properties:*

i) the homomorphisms T_λ are right inverses of $\overline\partial : L^{1,p}(\mathscr{A}_\lambda,\mathbb{C})\to L^p_{(0,1)}(\mathscr{A}_\lambda,\mathbb{C})$;

ii) the norms of T_λ, L_λ, and Q_λ, as well as L_λ^{-1} and Q_λ^{-1}, are uniformly bounded;

iii) the homomorphisms T_λ, L_λ, and Q_λ depend smoothly on $\lambda\neq 0$.

[3] The results presented in the lemma have been obtained jointly with S. Ivashkovich.

Proof. By the definition, the hyperbola metric $\frac{\mathrm{i}}{2}\left(1+\frac{|\lambda|^2}{|z^+|^4}\right)dz^+\wedge d\bar z^+$ on $\mathscr{A}_\lambda$ is the sum of the standard flat metrics $\frac{\mathrm{i}}{2}dz^+\wedge d\bar z^+$ and $\frac{\mathrm{i}}{2}dz^-\wedge d\bar z^-$. Note also that the function $\left(1+\frac{|\lambda|^2}{|z^+|^4}\right)$, restricted to the subannulus $\mathscr{A}^+_\lambda=\{|\lambda|^{1/2}<|z^+|<1\}\subset\mathscr{A}_\lambda$, takes values in the interval $[1,2]$. This implies that in every subannulus $\mathscr{A}^\pm_\lambda$ the metric $\frac{\mathrm{i}}{2}\left(1+\frac{|\lambda|^2}{|z|^4}\right)dz^+\wedge d\bar z^+$ is equivalent to the disc metric $\frac{\mathrm{i}}{2}dz^\pm\wedge d\bar z^\pm$. In particular, the norm $\|v\|_{L^{1,p}(\mathscr{A}_\lambda)}$ is equivalent to the norm

$$\left(\int_{|\lambda|<|z^+|^2<1}(|v|+|dv|)^p\,\frac{\mathrm{i}}{2}dz^+\wedge d\bar z^+ + \int_{|\lambda|<|z^-|^2<1}(|v|+|dv|)^p\,\frac{\mathrm{i}}{2}dz^-\wedge d\bar z^-\right)^{\frac{1}{p}}. \tag{5.4.11}$$

We start with construction of T_λ. For the discs $\Delta^\pm$ with the coordinates $z^\pm$ respectively we define $\tilde T^\pm : L^p_{(0,1)}(\Delta^\pm,\mathbb{C})\to L^{1,p}(\Delta^\pm,\mathbb{C})$ to be the Cauchy-Green operators, *i.e.*

$$\tilde T^+(\varphi^+)(z^+) := \frac{1}{2\pi\mathrm{i}}\int_{\zeta\in\Delta}\frac{d\zeta\wedge\varphi^+(\zeta)}{\zeta-z^+}$$

and similarly for $\tilde T^-$. Then we set

$$T^\pm(\varphi^\pm)(z^\pm) := \tilde T^\pm(\varphi^\pm)(z^\pm) - \tilde T^\pm(\varphi^\pm)(0)$$

So $T^\pm$ are normalizations of $\tilde T^\pm$ respectively to the condition $T^\pm(\varphi^\pm)(0)=0$. For a form $\varphi\in L^p_{(0,1)}(\mathscr{A}_\lambda,\mathbb{C})$ we denote by $\varphi^\pm(z^\pm)$ its restriction to $\mathscr{A}^\pm\subset\Delta^\pm$ extended by 0 to the whole discs $\Delta^\pm$, and set

$$T_\lambda(\varphi) := T^+(\varphi^+)(z^+) + T^-(\varphi^-)(z^-)$$

For the special case $\lambda=0$ this construction should be modified follows: Every form $\varphi\in L^p_{(0,1)}(\mathscr{A}_0,\mathbb{C})$ has two components $\varphi^\pm(z^\pm)$ corresponding to the decomposition

$$L^p_{(0,1)}(\mathscr{A}_0,\mathbb{C}) = L^p_{(0,1)}(\Delta^+,\mathbb{C})\oplus L^p_{(0,1)}(\Delta^-,\mathbb{C}),$$

and the operator T_0 transforms $\varphi^\pm$ into functions $f^\pm(z^\pm) := T^\pm_0(\varphi^\pm)(z^\pm)$ which satisfy $f^+(0)=f^-(0)=0$. Then $f := (f^+,f^-)\in L^{1,p}(\mathscr{A}_0,\mathbb{C})$. The desired properties of T_λ can be seen in a straightforward way.

Defining the operators L_λ, we recall the identification

$$\mathscr{H}(\mathscr{A}_0) = \{(g^+(z^+),g^-(z^-))\in\mathscr{H}(\Delta^+)\oplus\mathscr{H}(\Delta^-) : g^+(0)=g^-(0)\}.$$

We set $\mathsf{L}_0=\mathsf{Id}:\mathscr{H}(\mathscr{A}_0)\to\mathscr{H}(\mathscr{A}_0)$ and

$$\mathsf{L}_\lambda : g=(g^+(z^+),g^-(z^-))\in\mathscr{H}(\mathscr{A}_0)\mapsto g^+(z^+)+g^-(z^-)-g(0)$$

for $\lambda\neq0$. The desired properties of L_λ are obvious as well.

Note also that for $\lambda\neq0$ the inverse operator L^{-1}_λ essentially gives the Laurent decomposition of functions $g\in\mathscr{H}(\mathscr{A}_\lambda)$.

The definition of Q_λ is more subtle. For $\lambda\neq0\in\Delta(\varepsilon)$ we set

$$\rho_\lambda(r) := \frac{r^2-\frac{|\lambda|^2}{r^2}}{1-|\lambda|^2}\,. \tag{5.4.12}$$

Then every ρ_λ induces a diffeomorphism of the interval $[|\lambda|,1]$ onto $[-1,1]$, such that $[|\lambda|,|\lambda|^{1/2}]$ and $[|\lambda|^{1/2},1]$ are mapped onto the intervals $[-1,0]$ and $[0,1]$ respectively. The inverse map is given by

$$r = R_\lambda(\rho) = \sqrt{\frac{\rho(1-|\lambda|^2)+\sqrt{\rho^2(1-|\lambda|^2)^2+4|\lambda|^2}}{2}}\,. \tag{5.4.13}$$

For $\lambda \neq 0 \in \Delta(\varepsilon)$ we define the maps $\sigma^\pm_\lambda : Z(-1,1) = [-1,1]\times S^1 \to \mathscr{A}_\lambda$ which are given in the coordinates $z^\pm = r^\pm e^{\mathrm{i}\theta^\pm}$ by relations $r^\pm = R_\lambda(\rho)$ and $\theta^\pm = \theta$ respectively, so that

$$\sigma^\pm_\lambda : (\rho,\theta) \in Z(-1,1) \mapsto z^\pm = R_\lambda(\rho)e^{\mathrm{i}\theta} \in \mathscr{A}_\lambda.$$

Now we can explain the reason for the choice of (5.4.12), which at first glance probably seems rather wild. Our choice is made to insure that the pull-back by σ_λ of the natural volume form of $\mathscr{A}_\lambda$ is a constant multiple of the standard volume form on $Z(-1,1)$, *i.e.*

$$\sigma^*_\lambda\left(\left(1+\frac{|\lambda|^2}{(r^+)^4}\right)\frac{\mathrm{i}}{2}dz^+\wedge d\bar z^+\right) = (1-|\lambda|^2)d\rho\wedge d\theta. \tag{5.4.14}$$

Moreover, $1-|\lambda|^2$ remains uniformly bounded as λ varies in $\Delta(\varepsilon)$, so that the volume forms in the right hand side of (5.4.14) are equivalent.

The behavior of $\sigma^\pm_\lambda$ by λ close to 0, which is rather delicate, can be described as follows. Denote $Z^+ := Z(0,1)$ and $Z^- := Z(-1,0)$. Then for $\lambda \longrightarrow 0$ we have convergence of σ^+_λ on Z^+ to a map $\sigma^+_0 : Z^+ \to \mathscr{A}^+_0 = \Delta^+$, and resp. convergence of σ^-_λ on Z^- to $\sigma^-_0 : Z^- \to \mathscr{A}^-_0$, which are given by

$$\begin{aligned}\sigma^+_0 :&(\rho,\theta) \mapsto z^+ = \sqrt{\rho}\,e^{\mathrm{i}\theta},\\ \sigma^-_0 :&(\rho,\theta) \mapsto z^- = \sqrt{\rho}\,e^{\mathrm{i}\theta}.\end{aligned}$$

Observe also that we obtain the map $R_0(\rho) = \sqrt{\rho}$ in the limit of (5.4.13) as $\lambda \longrightarrow 0$. The convergence $\sigma^\pm_\lambda \longrightarrow \sigma^\pm_0$ is in the C^∞-sense in the interiors of $Z^\pm$, and in the C^0-topology up to boundary of $Z^\pm$.

On the other hand, there is no convergence of $\sigma^\pm_\lambda$ on $Z^\mp$. The topological reason for the absence of the convergence of $\sigma^\pm_\lambda|_{Z^\mp}$ is that, making with λ a small bypass around 0, we perform a *Dehn twist* with $\mathscr{A}_\lambda$.[4]

Now, we define Q_λ representing the components $\varphi^\pm$ of every $\varphi \in L^p_{(0,1)}(\mathscr{A}_\lambda,\mathbb{C}) = L^p_{(0,1)}(\Delta^+,\mathbb{C})\oplus L^p_{(0,1)}(\Delta^-,\mathbb{C})$ in the form $\varphi^\pm(z^\pm) = f^\pm(r^\pm,\theta^\pm)d\bar z^\pm$ with L^p-functions $f^\pm(r^\pm,\theta^\pm)$ and setting

$$Q_\lambda(\varphi) := \begin{cases} f^+\Big(R_\lambda\big((r^+)^2\big),\theta^+\Big)d\bar z^+ \ \text{ at the point } z^+ = r^+e^{\mathrm{i}\theta^+} \text{ with } r^+ \in [|\lambda|^{1/2},1];\\ f^-\Big(R_\lambda\big((r^-)^2\big),\theta^-\Big)d\bar z^- \ \text{ at the point } z^- = r^-e^{\mathrm{i}\theta^-} \text{ with } r^- \in [|\lambda|^{1/2},1].\end{cases} \tag{5.4.15}$$

Let us explain the meaning of the construction of Q_λ given in (5.4.15). The first point is that we essentially transform every form $\varphi \in L^p_{(0,1)}(\mathscr{A}_\lambda,\mathbb{C})$ into a function $f \in L^p(\mathscr{A}_\lambda,\mathbb{C})$. This is done by representing φ in the form $\varphi(z^+) = f^+(z^+)d\bar z^+$ on $\mathscr{A}^+_\lambda$ and in the form $\varphi(z^-) = f^-(z^-)d\bar z^-$ on $\mathscr{A}^-_\lambda$. Observe that we use different coordinates $z^\pm$ on the different parts $\mathscr{A}^\pm_\lambda$ of $\mathscr{A}_\lambda$. Independently of this, we obtain a well-defined L^p-function f on $\mathscr{A}_\lambda$, and hence a family of well-defined maps $F_\lambda : L^p_{(0,1)}(\mathscr{A}_\lambda,\mathbb{C}) \to L^p(\mathscr{A}_\lambda,\mathbb{C})$. It is not difficult to see that the F_λ are complex linear isomorphisms and that the operator norms $\|F_\lambda\|_{\mathsf{op}}$ and $\|F^{-1}_\lambda\|_{\mathsf{op}}$ are bounded uniformly in λ.

[4] The author is indebted to Bernd Siebert for this remark.

The second point is the observation that for the definition of a space $L^p(Y)$ only the involved measure μ on Y is essential. In particular, if a measurable map $g:(Y_1,\mu_1)\to(Y_2,\mu_2)$ induces an equivalence of measures, *i.e.* $g^*\mu_2=e^\rho(y)\mu_1$ for some bounded $\rho\in L^\infty(Y_1,\mu_1)$, then the induced map $g^*:L^p(Y_2,\mu_2)\to L^p(Y_1,\mu_1)$ is an isomorphism of Banach spaces. Thus our construction exploits the fact that the measures

$$(\sigma^\pm_\lambda)^*\left(\left(1+\frac{|\lambda|^2}{(r^\pm)^4}\right)\frac{\mathrm{i}}{2}dz^\pm\wedge d\bar z^\pm\right)$$

on $Z^\pm$ are equivalent.

Corollary 5.4.7. *The Banach spaces $L^{1,p}(\mathscr{A}_0,\mathbb{C}^n)$ and $L^{1,p}(\mathscr{A}_\lambda,\mathbb{C}^n)$ with $\lambda\neq0$ are isomorphic.*

Proof. One uses T_λ to split the exact sequences

$$0\longrightarrow\mathscr{H}(\mathscr{A}_\lambda,\mathbb{C}^n)\longrightarrow L^{1,p}(\mathscr{A}_\lambda,\mathbb{C}^n)\xrightarrow{\bar\partial}L^p_{(0,1)}(\mathscr{A}_\lambda,\mathbb{C}^n)\longrightarrow0$$

for $\lambda=0$ and $\lambda\neq0$. Then one applies L_λ and Q_λ to identify $\mathscr{H}(\mathscr{A}_0,\mathbb{C}^n)$ with $\mathscr{H}(\mathscr{A}_\lambda,\mathbb{C}^n)$ and respectively $L^p_{(0,1)}(\mathscr{A}_0,\mathbb{C}^n)$ with $L^p_{(0,1)}(\mathscr{A}_\lambda,\mathbb{C}^n)$. □

Let us explain now the main difficulty in the proof of *Theorem 5.4.1*. Several authors (see *e.g.* [Sie], [Li-T], [Ru]) have approached the *Gluing problem* by making an appropriate local imbedding $\mathscr{P}(\mathscr{A})\hookrightarrow L^{1,p}(\mathscr{A},\mathbb{C}^n)=\sqcup_{\lambda\in\Delta(\varepsilon)}L^{1,p}(\mathscr{A}_\lambda,\mathbb{C}^n)$ and showing that, roughly speaking, the $\bar\partial$-equation induces an operator which defines $\mathscr{P}(\mathscr{A})$ and has a certain (rather weak) smoothness property. This property is sufficient for showing that for any fixed $J\in\mathscr{J}$ the compactification $\overline{\mathscr{M}}_J$ has a well-defined "virtual fundamental class", the basic object to define in the theory of Gromov-Witten invariants.

The difficulty with this approach is that the smooth structure in $L^{1,p}(\mathscr{A},\mathbb{C}^n)$ given by the isomorphism $L^{1,p}(\mathscr{A},\mathbb{C}^n)\cong L^{1,p}(\mathscr{A}_0,\mathbb{C}^n)\times\Delta(\varepsilon)$ depends heavily on both the linear and the complex structures in $\mathbb{C}^n$. Moreover, one can show that for a generic smooth diffeomorphism $g:\mathbb{C}^2\to\mathbb{C}^2$ the induced map $u\in L^{1,p}(\mathscr{A},\mathbb{C}^n)\mapsto u\circ g\in L^{1,p}(\mathscr{A},\mathbb{C}^n)$ is not even Lipschitzian at generic $u\in L^{1,p}(\mathscr{A}_0,\mathbb{C}^n)$.

To the contrary, our approach of imbedding $\mathscr{P}(\mathscr{A})$ into $\mathscr{P}(V)\times\Delta(\varepsilon)$ by means of tracing the restriction $u|_V$ has the advantage that the smooth structure in $\mathscr{P}(V)$ is natural and canonical. Here we shall use *Lemma 5.4.6* rather in a different way: It serves for us as an approximative description of the behavior of the Gromov operator $D_{u,J,\lambda}$ as $\lambda\longrightarrow0$.

Definition 5.4.4. Let X be a manifold with a fixed symmetric connection ∇. Then for $\lambda\in\Delta(\varepsilon)$, $u\in L^{1,p}(\mathscr{A}_\lambda,X)$, and a C^1-smooth almost complex structure on X we denote by J_λ the complex structure on $\mathscr{A}_\lambda$ and by

$$D_{u,J,\lambda}:L^{1,p}(\mathscr{A}_\lambda,E_u)\to L^p_{(0,1)}(\mathscr{A}_\lambda,E_u)$$

the operator given by

$$D_{u,J,\lambda}(v):=\nabla v+J\circ\nabla v\circ J_\lambda+\frac12\big(\nabla_vJ\circ du\circ J_\lambda-J\circ\nabla_vJ\circ du\big).\tag{5.4.16}$$

Observe that in the definition of the space $L^p_{(0,1)}(\mathscr{A}_\lambda,E_u)$ we equip $E_u:=u^*TX$ with the structure u^*J. The construction of $D_{u,J,\lambda}$ is an extension of the definition of the Gromov operator, because for $(u,J)\in\mathscr{P}(\mathscr{A}_\lambda)$ the definition (5.4.16) coincides with the original one in (1.3.10).

Lemma 5.4.8. *Let $B \subset \mathbb{R}^{2n}$ be the ball and J^* a C^1-smooth almost complex structure in B. Then there exist constants $\varepsilon = \varepsilon(J) > 0$ and $C = C(J) < \infty$ such that for any almost complex structure J with*

$$\|J - J^*\|_{C^1(B)} \leqslant \varepsilon,$$

any $\lambda \in \Delta(\varepsilon)$, and any J-holomorphic map $u : \mathscr{A}_\lambda \to B(\frac{1}{2})$ with

$$\|du\|_{L^p(\mathscr{A}_\lambda)} \leqslant \varepsilon,$$

i) one has a uniform estimate

$$\|v\|_{L^{1,p}(\mathscr{A}_\lambda)} \leqslant C \cdot \big(\|v\|_{L^{1,p}(V)} + \|D_{u,J,\lambda} v\|_{L^p(\mathscr{A}_\lambda)}\big) \tag{5.4.17}$$

for any $v \in L^{1,p}(\mathscr{A}_\lambda, E_u)$;

ii) there exists an operator $T_{u,J,\lambda} : L^p_{(0,1)}(\mathscr{A}_\lambda, E_u) \to L^{1,p}(\mathscr{A}_\lambda, E_u)$ with $D_{u,J,\lambda} \circ T_{u,J,\lambda} = id$;

iii) moreover, the family of operators $T_{u,J,\lambda}$ depends continuously on (u, J, λ).

Proof. Recall that on each half-annulus $\mathscr{A}^\pm_\lambda$ the hyperbola metric (5.4.4) is equivalent to the (flat) disc metric $\frac{i}{2} dz^\pm \wedge \bar z^\pm$. Thus we can apply the Morrey estimate

$$\mathsf{diam}\,(u(\mathscr{A}^\pm_\lambda)) \leqslant C \cdot \|du\|_{L^p(\mathscr{A}^\pm_\lambda)}$$

which is uniform in λ. This gives

$$\mathsf{diam}\,(u(\mathscr{A}_\lambda)) \leqslant C \cdot \|du\|_{L^p(\mathscr{A}_\lambda)} \tag{5.4.18}$$

again uniformly in λ.

Consequently, $\mathsf{diam}\,(u(\mathscr{A}_\lambda))$ is sufficiently small. Let J_{st} be a linear complex structure in $\mathbb{R}^{2n}$ with coincides with J at some point $x_0 = u(z_0) \in u(\mathscr{A}_\lambda)$. Then $\|J \circ u - J_{\mathsf{st}}\|_{C^0(\mathscr{A}_\lambda)}$ is also small enough.

The canonical trivialization of the tangent bundle TB yields the canonical trivialization of $E_u = u^*TB$ and the identification $L^{1,p}(\mathscr{A}_\lambda, E_u) = L^{1,p}(\mathscr{A}_\lambda, \mathbb{C}^n)$. By *Corollary 5.4.7* we obtain a structure of a continuous Banach bundle on the union $\sqcup_{(u,\lambda) \in L^{1,p}(\mathscr{A},B)} L^{1,p}(\mathscr{A}_\lambda, E_u)$.

A similar identification for $L^p_{(0,1)}(\mathscr{A}_\lambda, E_u)$ requires some modification, since in the definition of this space one involves the complex structure $J_u := u^*J = J \circ u$. Therefore we must fix a *complex* isomorphism φ of (TB, J) with the trivial complex bundle (TB, J_{st}) over B. This means that φ is an $\mathbb{R}$-linear endomorphism of TB satisfying $\varphi \circ J = J_{\mathsf{st}} \circ \varphi$. Since J coincides with J_{st} at $x_0 = u(z_0)$ with $z_0 \in \mathscr{A}_\lambda$, we may also assume that $\|\varphi - \mathsf{Id}_{TB}\|_{C^0(u(\mathscr{A}_\lambda))}$ is small enough. Using φ, we obtain an isomorphism $\varphi_* : L^p_{(0,1)}(\mathscr{A}_\lambda, E_u) \xrightarrow{\cong} L^p_{(0,1)}(\mathscr{A}_\lambda, \mathbb{C}^n)$. Moreover, the norms $\|\varphi_*\|_{\mathsf{op}}$ and $\|\varphi_*^{-1}\|_{\mathsf{op}}$ are bounded uniformly in $\lambda \in \Delta(\varepsilon)$. Now the composition $\varphi_* \circ D_{u,J,\lambda}$ is a homomorphism between $L^{1,p}(\mathscr{A}_\lambda, \mathbb{C}^n)$ and $L^p_{(0,1)}(\mathscr{A}_\lambda, \mathbb{C}^n)$.

Using the trivialization $(u^*TB, u^*J_{\mathsf{st}})$ we define the operator

$$\overline\partial : L^{1,p}(\mathscr{A}_\lambda, \mathbb{C}^n) \to L^p_{(0,1)}(\mathscr{A}_\lambda, \mathbb{C}^n).$$

By construction, $\|\overline\partial - \varphi_* \circ D_{u,J,\lambda}\|_{\mathsf{op}}$ is small enough. Consequently, the restriction of $D_{u,J,\lambda}$ to the image of the operator T_λ constructed in *Lemma 5.4.6* is an isomorphism. Now the existence of $T_{u,J,\lambda}$ follows from the "closed graph theorem" of Banach.

Note that the choice of the point x_0 and the isomorphism φ used in the construction of $T_{u,J,\lambda}$ can be made continuous on (u, J, λ). This means that there exist families $x_0(u, J, \lambda)$ and $\varphi(u, J, \lambda)$ which depend continuously on (u, J, λ) and have the properties needed above. For example, one can set $x_0(u, J, \lambda) := u(1, \lambda)$, the image of the point $(1, \lambda) \in \mathscr{A}_\lambda$. For such a choice of $x_0(u, J, \lambda)$ and $\varphi(u, J, \lambda)$ the family $T_{u,J,\lambda}$ also depends continuously on (u, J, λ).

As a consequence of *ii)*, it is sufficient to prove (5.4.17) under the additional condition $D_{u,J,\lambda}v=0$. But then

$$\|v\|_{L^{1,p}(\mathscr{A}_\lambda)} \leqslant C_2\cdot\big(\|v\|_{L^{1,p}(V)}+\|\bar\partial v\|_{L^p(\mathscr{A}_\lambda)}\big)=C_2\cdot\big(\|v\|_{L^{1,p}(V)}+\|\bar\partial v-\varphi_*\circ D_{u,J,\lambda}v\|_{L^p(\mathscr{A}_\lambda)}\big)$$
$$\leqslant C_2\cdot\big(\|v\|_{L^{1,p}(V)}+\|\bar\partial-\varphi_*\circ D_{u,J,\lambda}\|_{\mathsf{op}}\cdot\|v\|_{L^{1,p}(\mathscr{A}_\lambda)}\big).$$

This yields (5.4.17) provided $C_2\cdot\|\bar\partial-\varphi_*\circ D_{u,J,\lambda}\|_{\mathsf{op}}\leqslant\frac12$.

Note once more that all the estimates in the proof are uniform in $\lambda\in\Delta(\varepsilon)$. □

Lemma 5.4.9. *The union $\sqcup_{\lambda\in\Delta(\varepsilon)}T\mathscr{P}(\mathscr{A}_\lambda)$ is a continuous locally trivial Banach bundle over $\mathscr{P}(\mathscr{A})$.*

Proof. We use the map pr_V to identify $\sqcup_{\lambda\in\Delta(\varepsilon)}\mathscr{P}(\mathscr{A}_\lambda)$ with its image in $\mathscr{P}(V)\times\Delta(\varepsilon)$. By *Lemma 5.4.8 iii)*, we can we consider $\sqcup_{\lambda\in\Delta(\varepsilon)}T\mathscr{P}(\mathscr{A}_\lambda)$ as subbundle of the bundle $T\mathscr{P}(V)\times\Delta(\varepsilon)$ over $\mathscr{P}(V)\times\Delta(\varepsilon)$. This defines the canonical topology on $\sqcup_{\lambda\in\Delta(\varepsilon)}T\mathscr{P}(\mathscr{A}_\lambda)$. Moreover, *Lemma 5.4.8 iii)* implies the claim at all (u,J,λ) with $\lambda\neq0$. Hence it remains to show the local triviality of the bundle in question in a neighborhood of a given $(u_0,J_0,0)\in\mathscr{P}(\mathscr{A}_0)$.

Consider first the special case when $\|du_0\|_{L^p(\mathscr{A}_0)}$ is small enough. Under this assumption, *Lemma 5.4.8* provides existence of a continuous family of splittings $L^{1,p}(\mathscr{A}_\lambda,E_u)=\mathsf{Ker}\,(D_{u,J,\lambda})\oplus\mathsf{Im}\,(T_{u,J,\lambda})$ defined in a neighborhood of $(u_0,J_0,0)$ in $\sqcup_{\lambda\in\Delta(\varepsilon)}\mathscr{P}(\mathscr{A}_\lambda)$. Moreover, it follows from the construction of $T_{u,J,\lambda}$ that the map $\bar\partial:\mathsf{Im}\,(T_{u,J,\lambda})\to L^p_{(0,1)}(\mathscr{A}_\lambda,\mathbb{C}^n)$ is an isomorphism. By *Corollary 5.4.7*, this implies local triviality of $\mathsf{Im}\,(T_{u,J,\lambda})$. In a similar way one shows that for a continuous family of isomorphisms $\varphi_{u,J,\lambda}:(TB,J)\to(TB,J_{\mathsf{st}})$ the operators

$$v\in\mathsf{Ker}\,(D_{u,J,\lambda})\subset L^{1,p}(\mathscr{A}_\lambda,E_u)\mapsto\big(\mathsf{Id}-T_\lambda\circ\bar\partial\big)\varphi_{u,J,\lambda}v\in\mathscr{H}(\mathscr{A}_\lambda,\mathbb{C}^n)\subset L^{1,p}(\mathscr{A}_\lambda,\mathbb{C}^n)$$

are isomorphisms. This gives the local triviality of $\mathsf{Ker}\,(D_{u,J,\lambda})$.

Now observe that the bundle

$$\{(v,\dot J)\in T_{(u,J)}\mathscr{P}(\mathscr{A}_\lambda):v=-T_{u,J,\lambda}(\dot J\circ du\circ J_\lambda)\}\tag{5.4.19}$$

is a complement to $\mathsf{Ker}\,(D_{u,J,\lambda})$ in $T_{(u,J)}\mathscr{P}(\mathscr{A}_\lambda)$. Since the bundle (5.4.19) is isomorphic to the lift of $T\mathscr{J}(B)$ onto $\mathscr{P}(\mathscr{A}_\lambda)$, it is locally trivial over the whole union $\sqcup_\lambda\mathscr{P}(\mathscr{A}_\lambda)$.

Turn back to the general case of the lemma. For a given $(u_0,J_0,0)\in\mathscr{P}(\mathscr{A}_0)$, find a radius $r>0$ such that for the subnode

$$\mathscr{A}'_0:=\{(z^+,z^-)\in\mathscr{A}_0:|z^+|<r,\ |z^-|<r\}=\Delta^+(r)\cup\Delta^-(r)$$

the norm $\|du_0\|_{L^p(\mathscr{A}'_0)}$ is small enough. Define

$$A^+:=\{z^+\in\Delta^+:\tfrac r2<|z^+|<1\},\qquad A^-:=\{z^-\in\Delta^-:\tfrac r2<|z^-|<1\},$$
$$\mathscr{A}'_\lambda:=\{(z^+,z^-)\in\mathscr{A}_\lambda:|z^+|<r,\ |z^-|<r\}.$$

and

$$V':=A^+\cup A^-,\qquad V'':=V'\cap\mathscr{A}'_0.$$

For $|\lambda|<\frac r2$ we identify $A^\pm$ with the corresponding subsets in $\mathscr{A}_\lambda$ by means of the coordinates $z^\pm$. Then for $|\lambda|<\frac r4$ A^+ is disjoint from A^-. This gives a family of coverings $\mathscr{A}_\lambda=\mathscr{A}'_\lambda\cup V'$ parameterized by λ with $|\lambda|<\frac r4$. Moreover, we can consider V' and V'' as constant (*i.e.* independent of λ) complex curves. Now, for any $(u,J,\lambda)\in\mathscr{P}(\mathscr{A})$ sufficiently close to $(u_0,J_0,0)$ we obtain the sequence

$$0\to T_{(u,J,\lambda)}\mathscr{P}(\mathscr{A}_\lambda)\xrightarrow{\alpha_{u,J,\lambda}}T_{(u,J,\lambda)}\mathscr{P}(V')\oplus T_{(u,J,\lambda)}\mathscr{P}(\mathscr{A}'_\lambda)\xrightarrow{\beta_{u,J,\lambda}}T_{(u,J,\lambda)}\mathscr{P}(V'')\to0,\tag{5.4.20}$$

where we set

$$\alpha_{u,J,\lambda}(v) := \big(v|_{V'}, v|_{\mathscr{A}'_\lambda}\big) \qquad \beta_{u,J,\lambda}(v,w) := v|_{V''} - w|_{V''}.$$

We claim that the sequence (5.4.20) is exact and splits. Since $\alpha_{u,J,\lambda}$ is obviously injective, it is sufficient to construct a right inverse to each $\beta_{u,J,\lambda}$, *i.e.* a homomorphism

$$\gamma_{u,J,\lambda} : T_{(u,J,\lambda)}\mathscr{P}(V'') \to T_{(u,J,\lambda)}\mathscr{P}(V') \oplus T_{(u,J,\lambda)}\mathscr{P}(\mathscr{A}'_\lambda)$$

such that $\beta_{u,J,\lambda} \circ \gamma_{u,J,\lambda} = \mathsf{Id}$. Furthermore, since $\beta_{u,J,\lambda}$ depends continuously on $(u,J,\lambda) \in \mathscr{P}(\mathscr{A})$ close to the $(u_0,J_0,0)$, it is sufficient to construct a right inverse only for the $\beta_{u_0,J_0,0}$.

Consider the restriction map between $L^{1,p}(\mathscr{A}_0)$ and $L^{1,p}(V'')$ induced by the imbedding $V'' \hookrightarrow \mathscr{A}_0$. It is well-known that this map admits a left inverse, *i.e.* a map $Q : L^{1,p}(V'') \to L^{1,p}(\mathscr{A}_0)$ such that $Q(v'')|_{V''} = v''$ for any $v'' \in L^{1,p}(V'')$. Let us use the same notation for the restriction of Q onto $T_{(u_0,J_0,0)}\mathscr{P}(V'') \subset L^{1,p}(V'')$. Recall that in *Lemma 5.4.8* we have constructed the operator $T_{u_0,J_0,0} : L^p_{(0,1)}(\mathscr{A}_0) \to L^{1,p}(\mathscr{A}_0)$ which is a right inverse to the operator $D_{u_0,J_0,0} : L^{1,p}(\mathscr{A}_0) \to L^p_{(0,1)}(\mathscr{A}_0)$. Denote by $\chi_{\mathscr{A}'_0} : L^p_{(0,1)}(\mathscr{A}_0, E_{u_0}) \to L^p_{(0,1)}(\mathscr{A}_0, E_{u_0})$ the operator given the multiplication on the characteristic function of the subset $\mathscr{A}'_0 \subset \mathscr{A}_0$. Now, we obtain a left inverse $\gamma_{u_0,J_0,0}$ to the operator $\beta_{u_0,J_0,0}$ setting

$$\gamma_{u_0,J_0,0}(v'') := \Big(\big(T_{u_0,J_0,0} \circ \chi_{\mathscr{A}'_0} \circ D_{u_0,J_0,0} \circ Q(v'')\big)|_{V'}, \big((T_{u_0,J_0,0} \circ \chi_{\mathscr{A}'_0} \circ D_{u_0,J_0,0} - \mathsf{Id}) \circ Q(v'')\big)|_{\mathscr{A}'_0}\Big)$$

for $v'' \in T_{(u_0,J_0,0)}\mathscr{P}(V'')$. Let us check that $\gamma_{u_0,J_0,0}$ has the desired properties. For a given $v'' \in T_{(u_0,J_0,0)}\mathscr{P}(V'')$, set $\tilde v'' := Q(v'')$. Then $D_{u_0,J_0,0}\tilde v''$ vanishes on V''. Consequently, $D_{u_0,J_0,0}\tilde v''$ is the sum of the forms $\varphi, \psi \in L^p_{(0,1)}(\mathscr{A}_0, E_{u_0})$ with the supports in $V' \backslash V''$ and $\mathscr{A}'_0 \backslash V''$ respectively. Moreover, $\psi = \chi_{\mathscr{A}'_0} \circ D_{u_0,J_0,0}\tilde v''$. From the relation $D_{u_0,J_0,0} \circ T_{u_0,J_0,0} = \mathsf{Id}$ we obtain

$$D_{u_0,J_0,0}\big(T_{u_0,J_0,0} \circ \chi_{\mathscr{A}'_0} \circ D_{u_0,J_0,0} \circ Q(v'')\big)|_{V'} = \big(\chi_{\mathscr{A}'_0} \circ D_{u_0,J_0,0} \circ Q(v'')\big)|_{V'} = 0.$$

This means that the first component $\big(T_{u_0,J_0,0} \circ \chi_{\mathscr{A}'_0} \circ D_{u_0,J_0,0} \circ Q(v'')\big)|_{V'}$ of $\gamma_{u_0,J_0,0}(v'')$ satisfies $D_{u_0,J_0,0}v = 0$. Thus the first component of $\gamma_{u_0,J_0,0}$ takes values in $T_{(u_0,J_0,0)}\mathscr{P}(V')$. In the same way one can show that the second component of $\gamma_{u_0,J_0,0}$ takes values in $T_{(u_0,J_0,0)}\mathscr{P}(\mathscr{A}'_0)$. Finally, it is obvious that the difference of the components of $\gamma_{u_0,J_0,0}(v'')$ is v''. The lemma follows. □

Proof of *Theorem 5.4.1*. Take some $(u_0, J_0) \in \mathscr{P}(\mathscr{A}_0)$. Using *Lemma 5.4.5 iii)*, we identify $\mathscr{P}(\mathscr{A}_0)$ with its image $\mathsf{pr}_V\big(\mathscr{P}(\mathscr{A}_0)\big) \subset \mathscr{P}(V)$ and $T_{(u_0,J_0)}\mathscr{P}(\mathscr{A}_0)$ with its image $\mathsf{pr}_V\big(T_{(u_0,J_0)}\mathscr{P}(\mathscr{A}_0)\big) \subset T_{(u_0,J_0)}\mathscr{P}(V)$.

We claim that there exists a closed complement to $\mathsf{pr}_V\big(T_{(u_0,J_0)}\mathscr{P}(\mathscr{A}_0)\big)$ in $T_{(u_0,J_0)}\mathscr{P}(V)$. To show this, let us fix a closed nodal complex curve C and an imbedding $\mathscr{A}_0 \hookrightarrow C$. Then there exists the unique open set $\widetilde V \subset C$ such that $\widetilde V \cap \mathscr{A}_0 = V$ and $\widetilde V \cup \mathscr{A}_0 = C$. Extend the bundle $E_{u_0} = u_0^*\mathbb{C}^n$ to an $L^{1,p}$-smooth complex bundle $\widetilde E$ over C. Also extend the operator $D_{u_0,J_0,0} : L^{1,p}(\mathscr{A}_0, E_{u_0}) \to L^p_{(0,1)}(\mathscr{A}_0, E_{u_0})$ to an operator $\widetilde D : L^{1,p}(C, \widetilde E) \to L^p_{(0,1)}(C, \widetilde E)$ of the form $\widetilde D = \overline\partial_{\widetilde E} + \widetilde R$ where $\overline\partial_{\widetilde E}$ is the Cauchy-Riemann operator corresponding to some holomorphic structure in $\widetilde E$ and $\widetilde R$ is a $\mathbb{C}$-antilinear L^p-integrable homomorphism between $\widetilde E$ and $\widetilde E \otimes \Lambda^{(0,1)}C$, *i.e.* $\widetilde R \in L^p\big(C, \overline{\mathsf{Hom}}_{\mathbb{C}}(\widetilde E, \widetilde E \otimes \Lambda^{(0,1)}C)\big)$. It is not difficult to show that the extensions $\widetilde E$ and $\widetilde D$ can be made in such a way that the operator $\widetilde D$ is an isomorphism.

In particular, there exists the inverse operator $\widetilde{T} : L^p_{(0,1)}(C, \widetilde{E}) \to L^{1,p}(C, \widetilde{E})$. For an open set $U \subset C$ define

$$\mathscr{H}(U) := \{v \in L^{1,p}(U, \widetilde{E}) : \widetilde{D}v = 0\}.$$

In particular, we have $\mathscr{H}(\mathscr{A}_0) = \mathsf{pr}_V\big(T_{(u_0,J_0)}\mathscr{P}(\mathscr{A}_0)\big)$ and similar identifications for V and V'. Define the operator $\beta : \mathscr{H}(\widetilde{V}) \oplus \mathscr{H}(\mathscr{A}_0) \to \mathscr{H}(V)$ setting $\beta(v, w) := v|_V - w|_V$. Then there exists an operator $\gamma : \mathscr{H}(V) \to \mathscr{H}(\widetilde{V}) \oplus \mathscr{H}(\mathscr{A}_0)$ which is left inverse to β. Indeed, the construction of the operator $\gamma_{u_0,J_0,0}$ made in the proof of *Lemma 5.4.9* can be applied with appropriate modifications. In particular, one uses $\widetilde{T}$ instead of $T_{u_0,J_0,0}$.

Observe that the kernel $\mathsf{Ker}\,(\beta)$ can be identified in a natural way with the kernel of $\widetilde{D} : L^{1,p}(C, \widetilde{E}) \to L^p_{(0,1)}(C, \widetilde{E})$. This implies that β is an isomorphism and γ its inverse. Consequently, every $v \in \mathscr{H}(V)$ can be uniquely represented in the form $v = v_1 + v_2$ such that v_1 extends to $\tilde{v}_1 \in \mathscr{H}(\widetilde{V})$ and v_1 extends to $\tilde{v}_2 \in \mathscr{H}(\mathscr{A}_0)$. The set of all $v_1 \in \mathscr{H}(V) = T_{u_0,J_0}\mathscr{P}(V)$ obtained in this way forms the desired closed compliment to $\mathsf{pr}_V\big(T_{(u_0,J_0)}\mathscr{P}(\mathscr{A}_0)\big)$ in $T_{(u_0,J_0)}\mathscr{P}(V)$.

The existence of such a complement implies that there exists a small ball $\mathscr{U} \subset \mathscr{P}(\mathscr{A}_0)$ centered at (u_0, J_0) and an open imbedding $F : \mathscr{U} \times \mathscr{B} \hookrightarrow \mathscr{P}(V)$ with the following properties:

- $\mathscr{B}$ is a small ball in a closed complement of $\mathsf{pr}_V\big(T_{(u_0,J_0)}\mathscr{P}(\mathscr{A}_0)\big)$ in $T_{\mathsf{pr}_V(u_0,J_0)}\mathscr{P}(V)$;
- the map F is a C^1-diffeomorphism onto image;
- the restricted map $F|_{\mathscr{U}\times\{0\}} : \mathscr{U} \to \mathsf{pr}_V(\mathscr{U}) \subset \mathsf{pr}_V(\mathscr{P}(\mathscr{A}_0)) \subset \mathscr{P}(V)$ coincides with pr_V.

In other words, $\mathscr{U} \times \mathscr{B}$ appears as a chart for $\mathscr{P}(V)$ in which $\mathsf{pr}_V(\mathscr{U}) = F(\mathscr{U} \times \{0\})$ is a linear subspace.

Now consider the natural projection $\pi_{\mathscr{U}} : \mathscr{U} \times \mathscr{B} \to \mathscr{U}$ restricted to $F^{-1}\big(\mathsf{pr}_V(\mathscr{P}(\mathscr{A}_\lambda))\big)$. By *Lemma 5.4.9* this is a C^1-diffeomorphism for all sufficiently small λ, provided the ball $\mathscr{U}$ is small enough. Denote by

$$\Phi_\lambda : \mathscr{U} \to \mathsf{pr}_V^{-1}\big(\mathsf{pr}_V(\mathscr{P}(\mathscr{A}_\lambda)) \cap F(\mathscr{U} \times \mathscr{B})\big) \subset \mathscr{P}(\mathscr{A}_\lambda)$$

the inverse map. We claim that the family Φ_λ, parameterized by $\lambda \in \Delta(\varepsilon')$ with a sufficiently small $\varepsilon' > 0$, has the properties stated in *Theorem 5.4.1*. In fact, it remains to check only the fact that the whole map

$$\Phi : \mathscr{U} \times \Delta(\varepsilon') \to \sqcup_{\lambda\in\Delta(\varepsilon')}\mathscr{P}(\mathscr{A}_\lambda)$$

is continuous. However, this follows from the construction of Φ. □

6. Symplectic isotopy problem in $\mathbb{CP}^2$

6.1. **Symplectic isotopy problem.** Let Σ, Σ' be two (connected) symplectically imbedded surfaces in a symplectic 4-fold (X, ω). Assume that they have the same homology class. Then they have the same genus, see *Lemma 1.1.2*. Thus one can ask whether or not there exists an isotopy $\{\Sigma_t\}_{t\in[0,1]}$ from Σ to Σ' such that all Σ_t are also symplectically imbedded. This is refered to as the symplectic isotopy problem.

The example of Fintushel and Stern [Fi-St] shows that there is no hope to obtain a results of this type in the case when $\langle c_1(X), [\Sigma]\rangle \leqslant 0$. Namely, they proved that under certain conditions on a symplectic 4-fold (X, ω) there exists an infinite collection of symplectic imbeddings $\Sigma_i \hookrightarrow X$, such that Σ_i represent the same homology class $[C] \in \mathsf{H}_2(X, \mathbb{Z})$ but are pairwise non-isotopic, even smoothly. Moreover, the class of symplectic

4-folds with these conditions is sufficiently wide, so that one has enough examples of this type.

On the other hand, *Theorem 4.5.1* hints that a satisfactory solution for the symplectic isotopy problem in the case $\langle c_1(X), [C]\rangle \geqslant 1$ is possible. We state the problem in a more precise form.

Conjecture 1. *(Symplectic isotopy problem). Let (X,ω) be a compact symplectic 4-dimensional manifold and $[C] \in \mathsf{H}_2(X,\mathbb{Z})$ a homology class with $\langle c_1(X), [C]\rangle \geqslant 1$. Then every two symplectically immersed surfaces Σ and Σ' in the class $[C]$ are symplectically isotopic provided they have the same genus g and the only singularities are positive nodal points.*

Recall, there exists a complete classification of compact symplectic 4-folds X which come in question. Namely, **Corollary 1.5** in [McD-Sa-3], claims

Proposition 6.1.1. *Let X be a symplectic manifold and $\Sigma \subset X$ a symplectically imbedded surface which is not an exceptional sphere. Then X is the blow-up of a rational or ruled manifold.*

The complete description of possible symplectic structures on such X was done in [McD-4], [La-McD], and [McD-Sa-3], see also [Li-Liu], [Liu].

As the main result of this paper we give a positive solution of the symplectic isotopy problem for imbeddings of low degree in $\mathbb{CP}^2$.

Theorem 6.1.2. *Any two symplectically imbedded surfaces Σ, $\Sigma' \subset \mathbb{CP}^2$ of the same degree $d \leqslant 6$ are symplectically isotopic.*

The case $d = 1$ and 2 of the theorem has been proven by Gromov in [Gro], the case $d = 3$ by Sikorav [Sk-3].

In this connection a result of S. Finashin about (non-symplectic) isotopy problem in $\mathbb{CP}^2$ should be mentioned. He proves in [Fin] that for any even degree $d = 2k \geqslant 6$ there exist infinitely many isotopy classes of imbedded real surfaces in $\mathbb{CP}^2$ having the degree d and the genus g given by the genus formula, *i.e.* $g = \frac{(d-1)(d-2)}{2}$. Note that *Theorem 6.1.2* claims that for $d = 6$ only one of these isotopy classes is realizable by a *symplectic* imbedding.

Let us explain main ideas of the proof of *Theorem 6.1.2*. First we observe that existence of a symplectic isotopy $\{\Sigma_t\}_{t\in[0,1]}$ between surfaces Σ, Σ' in a symplectic manifold (X,ω) implies existence of an "accompanying" homotopy $\{J_t\}_{t\in[0,1]}$ of tame almost complex structures, such that the imbeddings $\Sigma_t \hookrightarrow X$ are J_t-holomorphic. Conversely a homotopy of ω-tame J_t-holomorphic imbeddings is necessarily a symplectic isotopy. So given Σ_0 and Σ_1, the natural thing to do is to outfit them with compatible structures J_0 and J_1, take a generic curve J_t and attempt to find appropriate liftings Σ_t. We do this using the following theorem of Harris [Ha] for an intermediate construction.

Proposition 6.1.3. *Any two irreducible nodal algebraic curves C_0 and C_1 in $\mathbb{CP}^2$ of the same degree d and the same geometric genus g are holomorphically isotopic, i.e. can be connected by an isotopy $\{C_t\}_{t\in[0,1]}$ consisting of nodal algebraic curves.*

By this result, in order to construct the symplectic isotopy, it is enough to construct a lifting as above for the case where J_1 is the standard integrable structure on $\mathbb{CP}^2$ and Σ_1 is some smooth algebraic curve.

Obviously, *Theorem 4.5.1* would imply existence of symplectic isotopy if we could show that for a generic path $\{J_t\}_{t\in[0,1]}$ the moduli space $\mathscr{M}_{J_t}$ is non-empty. An obstruction to

this is the fact that the projection $\pi_{\mathscr{J}} : \mathscr{M} \to \mathscr{J}$ is not proper. This means that we must understand the structure of the total moduli space $\mathscr{M}$ "at infinity". In *Paragraph 5.2* we have constructed a completion $\overline{\mathscr{M}}$ of $\mathscr{M}$ and eqquiped it with a natural stratification such that every stratum is a smooth Banach manifold. In particular, the transversality technique developed in *Section 2* can be applied to every such stratum.

The next idea in the proof of *Theorem 6.1.2* is to construct a path $\tilde{\gamma}_t := (C_t, J_t) \in \overline{\mathscr{M}}$ which goes piecewisely along some strata and which can be "pushed" into the "main stratum" $\mathscr{M}$ yielding the desired isotopy (Σ_t, J_t). The main difficulty in realization this idea is to ensure that pushing $\tilde{\gamma}_t$ into $\mathscr{M}$ we still remain in the same connected component of $\mathscr{M}$ so that the symplectic isotopy class is preserved. This means that we are interested in describing possible connected components of $\mathscr{M}$ in a neighborhood of a given curve $(C^*, J^*) \in \overline{\mathscr{M}}$. Moreover, the positive solution of a symplectic isotopy problem would follow immediately from the fact that locally exactly one such component exists. Indeed, it would be then sufficient to construct *any* path $\tilde{\gamma}_t := (C_t, J_t) \in \overline{\mathscr{M}}$ connecting Σ and Σ'. But existence of such a path follows easily from *Theorem 4.5.1* in the case $c_1(X)[C] > 0$.

The result of *Theorem 6.1.2* is obtained via the proof of the local uniqueness of such a component of $\mathscr{M}$ near a given $(C^*, J^*) \in \overline{\mathscr{M}}$ in the special case when C^* contains no multiple components. The restriction $d \leqslant 6$ in the theorem comes from the fact that in this case it is possible to avoid the appearance of multiple components in C^*. We are able to do so by demanding that the pseudoholomorphic curves Σ_t in the isotopy path pass through fixed generic $3d-1$ points on $X = \mathbb{CP}^2$. Note that the number $3d-1$ is the maximal possible in *Theorem 4.5.3*.

6.2. **Local symplectic isotopy problem.** As we have explained in the previous paragraph, we are interested in the possible symplectic isotopy classes of pseudoholomorphic curves C in a neighborhood of a given singular curve C^* with no multiple components. The main difficulty in this case is, of course, to understand the local behavior of curves C near singular points of C^*. In this way we come to the following question.

The Local Symplectic Isotopy Problem. *Let B be the unit ball in $\mathbb{R}^4$ equipped with the standard symplectic structure ω_{st}, J^* an ω_{st}-tame almost complex structure, and $C^* \subset B$ a connected J^*-holomorphic curve in B with a unique isolated singularity at $0 \in B$ and without multiple components. Describe the possible symplectic isotopy classes of curves C in B which lie sufficiently close to C^* with respect to the cycle topology and which have prescribed singularities, e.g. prescribed number of nodes and ordinary cusps.*

We start with a construction of certain symplectic isotopy classes of nodal pseudoholomorphic curves. For C^* as above, let $C^* = \cup_i C_i^*$ be the decomposition into irreducible components. Then there exist J^*-holomorphic parameterizations $u_i^* : S_i \to B$, $u_i^*(S_i) = C_i^*$. Shrinking C_i^*, if needed, we may assume that all S_i are compact and smooth boundaries ∂S_i, each consisting of finitely many circles. Note that the images of the boundary circles are imbedded in B and mutually disjoint. Further, we can also suppose that u_i^* are $L^{1,p}$-smooth up to boundaries ∂S_i. Set $S := \sqcup S_i$ and define $u^* : S \to B$ by $u^*|_{S_i} := u_i^*$. Denote by J_S^* the complex structure on S induced by $u^* : S \to B$ from C^*.

Lemma 6.2.1. *i) The set $\mathscr{P}(S,B)_{\mathsf{nod}}$ of those $(u, J_S, J) \in \mathscr{P}(S,B)$ for which the map $u : S \to B$ is an immersion and the singularities of the image $C := u(S)$ are only nodal points is open and dense in $\mathscr{P}(S,B)$ and is connected;*

ii) For (u', J'_S, J') and $(u'', J''_S, J'') \in \mathscr{P}(S,B)_{\mathsf{nod}}$, sufficiently close to $(u^*, J_S^*, J^*) \in \mathscr{P}(S,B)$, the pseudoholomorphic curves $C' := u'(S)$ and $C' := u'(S)$ are symplectically isotopic;

iii) For a fixed $J_S \in \mathscr{J}_S$ and $J \in \mathscr{J}(B)$, the subspace of nodal curves in each of the spaces $\mathscr{P}(S;B,J)$, $\mathscr{P}(S,J_S;B)$, and $\mathscr{P}(S,J_S;B,J)$ is open and dense in thecorresponding space.

Proof. By results of *Section 3*, the complement to $\mathscr{P}(S,B)_{\mathsf{nod}}$ in $\mathscr{P}(S,B)$ is closed and consists of submanifolds of real codimension at least 2. This shows *i)* and implies *ii)* . Part *iii)* is obtained similarly. □

Definition 6.2.1. In the situation of *Lemma 6.2.1*, we call $C_{\mathsf{nod}} = u(S)$ a *maximal nodal deformation of* C^* and the number δ of nodes on C_{nod} the *nodal number of* C^* at the singular point $0 \in C^*$. In other words, a maximal nodal deformation is a nodal pseudoholomorphic curve obtained from $C^* = u^*(S)$ by a (sufficiently small) generic deformation of the parameterization map $u^* : S \to X$, $C^* = u^*(S)$.

Further, a *canonical smoothing of* C^* is a J^*-holomorphic curve $C^\dagger$ obtained from a maximal nodal nodal deformation C_{nod} by smoothing of all nodes. We use the notion of canonical smoothing for both the construction and the resulting curve. Further, we shall always assume that a canonical smoothing $C^\dagger$ is sufficiently close to C^* with respect to the cycle topology.

It follows immediately from *Lemma 6.2.1* that the symplectic isotopy class of a canonical smoothing of C^* is well-defined.

Proposition 6.2.2. *Any two curves $C_1^\dagger$ and $C_2^\dagger$ obtained from C^* by the construction of canonical smoothing are symplectically isotopic. Moreover, such an isotopy can be be carried out sufficiently close to the identity map.*

Note that the number $\delta(C_{\mathsf{nod}})$ of nodes on a maximal nodal deformation C_{nod} of C^* equals the nodal number $\delta(0, C^*)$ of C^* at 0. Observe also that one can smooth *some* number of nodes on C_{nod} producing further symplectic isotopy classes. It is easy to show that these new classes are determined by the set of the nodes on C_{nod} which are smoothed. We conjecture that these are all possible symplectic isotopy classes of nodal curves in a neighborhood of C^* with respect to the cycle topology.

Conjecture 2. *(Local symplectic isotopy problem for nodal curves). Let J^* be a C^2-smooth ω_{st}-tame almost complex structure in $B \subset \mathbb{R}^4$ and $C^* \subset B$ a J^*-holomorphic curve with a unique isolated singular point at $0 \in B$ and without multiple components. Assume that J is an almost complex structure in B which is $C^{0,\alpha}$-smooth for $\alpha > 0$ and sufficiently close to J^* with respect to the $C^{0,\alpha}$-topology.*

Then any nodal J-holomorphic curve C sufficiently close to C^* with respect to the cycle topology is symplectically isotopic to a J^*-holomorphic curve obtained from a maximal nodal deformation C_{nod} of C^* by smoothing some number of nodes on C_{nod}.

We give a proof the conjecture for the case of *imbedded* curves. Observe that here we have only one candidate, namely the canonical smoothing.

Theorem 6.2.3. *In the situation described in Conjecture 2, let $C^\dagger$ be J^*-holomorphic curve obtained by the canonical smoothing of C^*.*

Let J be an almost complex structure on B sufficiently close to J^* with respect to the $C^{0,\alpha}$-topology and C an imbedded J-holomorphic C sufficiently close to C^* with respect to the cycle topology. Then there exist a homotopy J_t which is C^0-sufficiently close to J^* and connects J^* with J, and an isotopy C_t of J_t-holomorphic curves which connects $C^\dagger$ with C and is sufficiently close to C^* with respect to the cycle topology.

The proof will be given after some preparatory results. We shall always assume that the hypotheses of the theorem are fulfilled. Denote by S the real surface parameterizing C. In other words S is the curve C, considered as real oriented surface without complex structure.

Our first observation is that the theorem holds in the case when C^* and the approximating curve C are holomorphic in the usual sense. The result is well-known, see *e.g.* [Mil]. Its proof is based on the main advantage of the holomorphic case: the fact that one can represent a holomorphic curve as the zero divisor of a holomorphic function.

Lemma 6.2.4. *Let f^* be a holomorphic function in the ball B in $\mathbb{C}^2$ whose zero divisor is a holomorphic curve C^* with a single singular point at $0 \in B$ and without multiple components. Assume that f^* and C^* are sufficiently smooth also at the boundary ∂B. Then*

- *i) a canonical smoothing C is obtained as the zero divisor of a sufficiently small perturbation f of f^*;*
- *ii) for two generic sufficiently small perturbations f_1 and f_2 of f^* their zero divisors C_0 and C_1 are non-singular and* holomorphically isotopic, *i.e. can be connected by a homotopy consisting of holomorphic non-singular curves C_t.*

Denote by δ^* the nodal number of C^* at $0 \in C^*$. We may assume inductively that the claim of *Theorem 6.2.3* holds for all curves C' which satisfy the hypotheses of the theorem but have the nodal number $\delta(C')$ at $0 \in C'$ which is strictly less than δ^*. Further, we assume that $\delta^* \geqslant 2$, since otherwise $\delta^* = 1$ and $0 \in C^*$ is a nodal point, the case covered by *Paragraph 5.4*.

Recall that by the theorem of Micallef and White (see *Lemma 1.2.1*) in a neighborhood of $0 \in B$ there exists a C^1-diffeomorphism φ of $B \subset \mathbb{R}^4$ such that $\varphi(0) = 0$, $\varphi_*(J^*(0)) = J_{\mathsf{st}}$, the standard complex structure in $\mathbb{R}^4 = \mathbb{C}^2$, and such that $\varphi(C^*)$ is a J_{st}-holomorphic curve. Obviously, we may also assume that $d\varphi : T_0B \to T_0B$ is the identity map. This means that the form $\varphi_*\omega_{\mathsf{st}}$ coincides with ω_{st} at $0 \in B$, $\varphi_*(\omega_{\mathsf{st}})|_{T_0B} = \omega_{\mathsf{st}}$, and similarly $\varphi_*(J^*(0)) = J^*(0) = J_{\mathsf{st}}$. Consequently, $\varphi_*(J^*)$ is ω_{st}-tame in a sufficiently small ball $B(r)$, $r \ll 1$. Let us fix such a radius r.

Moreover, since $C^* \subset B$ is imbedded outside 0, we can additionally assume that φ is *smooth* outside $0 \in B$.

Below, we translate the original situation by means of such φ and work with a *holomorphic* curve $\varphi(C^*) \cap B(r)$. This leads to the difficulty that $\varphi_*(J^*)$ is apriori only *continuous* at $0 \in B(r)$. This requires an additional control on the behavior of pairs $(C, J) \in \mathscr{P}(B)$ approximating (C^*, J^*).

Lemma 6.2.5. *Let $(u_n, J_{S,n}, J_n) \in \mathscr{P}(S, B)$ be a sequence such that J_n converges to J^* in the $C^{0,\alpha}$-topology with $0 < \alpha < 1$, and $C_n := u_n(S)$ converges to C^* with respect to the cycle topology and with respect to the $L^{1,p}$-topology near boundary $\partial C_n = u_n(\partial S)$. Further, let $\varphi : B \to B$ be the diffeomorphism introduced above. Then for all sufficiently big n*

i) $u_n : S \to B$ is an imbedding;

ii) there exists a sequence J^*_n of C^ℓ-smooth almost complex structures in B such that

- *u_n are $(J_{S,n}, J^*_n)$-holomorphic;*
- *$\varphi_*(J^*_n)$ converges to J_{st} in the C^0-topology in $B(r)$ and in the $C^{0,\alpha}$-topology outside $0 \in B(r)$.*

Proof. The first part follows from *Lemma 1.2.3*, applied to a smaller ball $B(\rho)$, $\rho < 1$, and curves $C^* \cap B(\rho)$, $C_n \cap B(\rho)$.

Define $J^\sharp$ as the pull-back of J_{st} with respect to φ, $J^\sharp := \varphi^*(J_{\mathsf{st}})$. Then the second part is equivalent to the convergence $J^*_n \longrightarrow J^\sharp$ in the appropriate topology.

Fix some sufficiently small $\varepsilon > 0$. Since $J^*(0) = J_{\mathsf{st}} = J^\sharp(0)$, there exists a positive radius $\rho \ll r$ such that $\| J^* - J^\sharp \|_{C^0(B(\rho))} < \varepsilon$. This implies that $\| J_n - J^\sharp \|_{C^0(B(\rho))} < \varepsilon$ for all sufficiently big n.

Now observe that in $B \backslash B(\rho)$ we have the $C^{1,\alpha}$ convergence $C_n \longrightarrow C^*$. In particular, in $B \backslash B(\rho)$ we have $C^{0,\alpha}$-convergence of tangent bundles $TC_n \longrightarrow TC^*$. This implies that for $n \gg 1$ we can extend every J_n from $B(\rho)$ to $B(r)$ as a C^ℓ-smooth structure J^*_n which is defined *along* C_n and obeys the estimate

$$\| J^*_n - J^\sharp \|_{C^0(C_n \cap B(r))} < \varepsilon, \tag{6.2.1a}$$

$$\| J^*_n - J^\sharp \|_{C^{0,\alpha}(C_n \cap (B(r) \backslash B(2\rho)))} < \varepsilon. \tag{6.2.1b}$$

Finally we extend the constructed J^*_n from $C_n \cup B(\rho)$ to the whole ball B preserving the estimates (6.2.1). □

Remark. In fact, below we shall merely make use of the weaker C^0-convergence $\varphi_*(J^*_n) \to J_{\mathsf{st}}$. The Hölder $C^{0,\alpha}$-convergence $J_n \to J^*$ was used only to provide the C^0-convergence of tangent bundles $TC_n \to TC^*$ outside $0 \in C^*$. In particular, it would be sufficient to have only C^0-convergence $J_n \to J^*$ in B and the $C^{0,\alpha}$-convergence outside $0 \in B$. On the other hand, in the case when the convergence $J_n \to J^*$ is better, say in the C^ℓ-topology with non-integer $\ell > 1$, we could achieve just as as well the C^ℓ-convergence in $B(r)$ outside 0.

Lemma 6.2.5 insures that we can reduce the problem to the case when C^* is holomorphic in the usual sense, *i.e.* with respect to the structure J_{st}. Further, observe that for the proof of *Theorem 6.2.3* it is sufficient to show that for any sequence $(u_n, J_{S,n}, J_n)$ satisfying the hypotheses of *Lemma 6.2.5* the curves $C_n := u_n(S)$ are symplectically isotopic to $C^\dagger$ for $n \gg 1$. An equivalent problem is to show that $\varphi(C_n)$ are symplectically isotopic to $\varphi(C^\dagger)$ in $B(r)$. Thus we can replace our initial objects by their φ-images in $B(r)$. For the sake of simplicity we maintain the original notations for these new objects, *e.g.* B for $B(r)$, C^* and C_n for respectively $\varphi(C^*) \cap B(r)$ and $\varphi(C_n) \cap B(r)$, J^* and J_n for respectively $\varphi_*(J^*)|_{B(r)}$ and $\varphi_*(J_n)|_{B(r)}$, and so on. On the other hand, J_{st} and ω_{st} remain the standard structures in B. Observe that now we have the weaker C^0-convergence $J_n \longrightarrow J^*$.

Imbed B in $\mathbb{CP}^2$ in the standard way so that J_{st} becomes the standard integrable structure, still denoted by J_{st}. Then we can extend ω to a global symplectic form on $\mathbb{CP}^2$ taming J_{st}. We maintain the notation ω for this extension.

We claim that C^* also extends to $\mathbb{CP}^2$ as a compact closed pseudoholomorphic curve. Moreover, we claim that there exists an extension $\tilde{C}^*$ with the following properties

- all irreducible components of $\tilde{C}^*$ are rational, *i.e.* parameterized by the sphere S^2;
- except for the original singularity at $0 \in \tilde{C}^*$, all new singularities are only nodal points.

Indeed, every irreducible component of $C^* \subset B$ is parameterized by a holomorphic map $u_i = u_i(z) : \Delta \to B$ with $u_i(0) = 0$. For every $u_i(z)$ we take the Taylor polynomials $u_i^{(d)}(z)$ of degree d chosen sufficiently high to satisfy the following conditions:

- every $u_i^{(d)}(z)$ is non-multiple;
- the images $u_i^{(d)}(\Delta)$ are pairwise distinct holomorphic discs.

Then every $u_i^{(d)}(z)$ can be considered as an algebraic map f_i from $\mathbb{CP}^1 = S^2$ to $\mathbb{CP}^2$. Making a generic perturbation of f_i outside B, we obtain desired curve $\tilde{C}^* \subset \mathbb{CP}^2$ as the union of the images $\tilde{f}_i(S^2)$ of the perturbed maps. Observe that d appears as the degree of every component $\tilde{C}_i^* := \tilde{f}_i(S^2)$.

Lemma 6.2.6. *There exist an almost complex structure $\tilde{J}^*$ and points x_α on $\tilde{C}^*$ satisfying the following conditions:*

(a) the points x_α are pairwise distinct, and there are exactly $3d-1$ of them on every component $\tilde{C}_i^*$;
(b) $\tilde{J}^*$ is C^ℓ-smooth and ω-tame, $\tilde{C}^*$ is $\tilde{J}^*$-holomorphic, and $\tilde{J}^*$ coincides with J^* on B;
(c) any $\tilde{J}^*$-holomorphic curve C' which
  - *passes through the fixed points x_α;*
  - *is sufficiently close to $\tilde{C}^*$ with respect to the cycle topology;*
  - *has the same number of singular points as $\tilde{C}^*$;*

  $*$ *has a singular point $x' \in C'$ with the nodal number δ^* at x'*

  must coincide with $\tilde{C}^*$.

The last property asserts that every pseudoholomorphic curve $C' \neq \tilde{C}^*$ with the properties *(c)* except $(*)$ has simpler singularities than $\tilde{C}^*$. So the induction assumption can be applied to such a C'.

Proof. We use the results of *Sections 2* and *3*. Fix non-singular points x_α on $\tilde{C}^*$ such that condition *(a)* is fulfilled. Let $\boldsymbol{x}_i$ be the $(3d-1)$-tuple of the points lying on the component $\tilde{C}_i^*$. Denote by $\mathscr{M}'$ the space of pairs (C', J'), where $J' \in \mathscr{J}$ and C' is J'-holomorphic curve C' satisfying properties *(c)* except $(*)$. Then by the genus formula (1.2.1) any such curve C' has only rational irreducible components C_i', the number of which is the same as for $\tilde{C}^*$, and the degree of every component C_i' is d. This means that $\mathscr{M}'$ is the fiber product of the spaces $\mathscr{M}(S^2, \mathbb{CP}^2, d, \boldsymbol{x}_i)$ of rational pseudoholomorphic curves of degree d in $\mathbb{CP}^2$ passing through $\boldsymbol{x}_i$. The product is taken over the space $\mathscr{J}$ of almost complex structure in $\mathbb{CP}^2$. By the transversality technique of *Section 2*, the space $\mathscr{M}'$ a Banach manifold. To compute the Fredholm index of the natural projection $\pi'_{\mathscr{J}} : \mathscr{M}' \to \mathscr{J}$ observe that the expected dimension of rational J-holomorphic curves in $\mathbb{CP}^2$ of degree d passing through $3d-1$ fixed distinct points is 0. This implies that the index of the projection $\pi'_{\mathscr{J}} : \mathscr{M}' \to \mathscr{J}$ is also 0.

Further, by results of *Section 3* the condition $(*)$ defines a proper C^ℓ-smooth submanifold $\mathscr{M}^*$ in $\mathscr{M}'$ of finite codimension, say m. Consequently, the index of the corresponding projection $\pi^*_{\mathscr{J}} := \pi'_{\mathscr{J}}|_{\mathscr{M}^*} : \mathscr{M}^* \to \mathscr{J}$ is negative. Using the transversality technique of *Section 2* we can construct a C^ℓ-smooth submanifold $Y \subset \mathscr{M}^*$ of dimension m such that

- $(\tilde{C}^*, \tilde{J}^*) \in Y$ for some $\tilde{J}^*$ obeying the condition *(b)* of the lemma;
- Y is transversal to $\mathscr{M}^*$;
- the restricted projection $\pi'_{\mathscr{J}}|_Y : Y \to \mathscr{J}$ is an imbedding.

Then $(\tilde{C}^*, \tilde{J}^*)$ is an isolated point of the intersection $Y \cap \mathscr{M}^*$. But this means that $\tilde{J}^*$ has the desired properties. □

Below we shall need a property which is a bit sharper than *(c)* in *Lemma 6.2.6*. Roughly speaking, it claims that one can recover a pseudoholomorphic curve C in B knowing its part $\big(\overline{B} \backslash B(\tfrac{1}{2})\big) \cap C$.

Definition 6.2.2. Denote by A the spherical annulus $\overline{B}\backslash B(\frac{1}{2})$. It is a closed subset of the closed unit ball $\overline{B} \subset \mathbb{CP}^2$. For closed subsets $Y_1, Y_2 \subset \mathbb{CP}^2$ we denote by $\mathsf{dist}_A(Y_1, Y_2)$ the Hausdorff distance between $Y_1 \cap A$ and $Y_2 \cap A$,

$$\mathsf{dist}_A(Y_1, Y_2) := \mathsf{dist}\,(Y_1 \cap A, Y_2 \cap A),$$

if both $Y_1 \cap A$ and $Y_2 \cap A$ are non-empty. The standard distance function in $\mathbb{CP}^2$ is used as the base. If exactly one of the set $Y_i \cap A$ is empty, we set $\mathsf{dist}_A(Y_1, Y_2) := \mathsf{diam}\,(\mathbb{CP}^2)$. If $Y_1 \cap A = Y_2 \cap A = \varnothing$, we define $\mathsf{dist}_A(Y_1, Y_2) := 0$. We call dist_A the A-*distance*.

It is easy to see that dist_A is only a pseudo-distance function, *i.e.* it is non-negative, symmetric, and has the triangle inequality property, but does not distinguish all closed subsets $Y_1 \neq Y_2 \subset \mathbb{CP}^2$ in general. It turns out that it induces the cycle topology on the set of pseudoholomorphic curves lying in a sufficiently small dist_A-neighborhood of $\tilde{C}^*$ provided only C^1-smooth almost complex structures J are used. More precise statement is given in

Lemma 6.2.7. *There exists an $\varepsilon > 0$ with the following property.*

Let $J \in \mathscr{J}$ be a C^1-smooth almost complex structure which satisfies the condition $\|J - J^*\|_{C^0(\mathbb{CP}^2)} \leqslant \varepsilon$ and C a J-holomorphic curve which is homologous to $\tilde{C}^*$ and satisfies the condition $\mathsf{dist}_A(C, \tilde{C}^*) \leqslant \varepsilon$. Then for any sequence J_n of continuous almost complex structures J_n converging to J in the C^0-topology, $\|J_n - J\|_{C^0(\mathbb{CP}^2)} \longrightarrow 0$, and any sequence of J_n-holomorphic curves C_n the condition $\mathsf{dist}_A(C_n, C) \longrightarrow 0$ implies that C_n converges to C in the cycle topology.

Proof. Consider a sequence of almost complex structures J_n in $\mathbb{CP}^2$ which converges to J^* in the C^0-topology, and a sequence C_n of closed J_n-holomorphic curves homologous to C^*, for which $\lim \mathsf{dist}_A(C_n, \tilde{C}^*) = 0$. Then J_n are ω_{st}-tame for all $n \gg 1$. Hence we can apply the Gromov compactness theorem (see *Theorem 5.1.1*). This means that some subsequence, still denoted C_n, converges to a J^*-holomorphic curve C^+ with respect to the cycle topology. The condition $\lim \mathsf{dist}_A(C_n, \tilde{C}^*) = 0$ implies that $\mathsf{dist}_A(C^+, \tilde{C}^*) = 0$, which means that $\tilde{C}^* \cap A = C^+ \cap A$.

Observe now that by the construction of $\tilde{C}^*$ every irreducible component $\tilde{C}^*_i$ of $\tilde{C}^*$ meets the interior $\mathsf{Int}(A)$ of A. By the unique continuation property of pseudoholomorphic curves proven in *Lemma 1.2.5 ii)*, every component $\tilde{C}^*_i$ is contained in C^+. Thus $\tilde{C}^* \subset C^+$. Since C^+ is homologous to $\tilde{C}^*$, we must have equality $\tilde{C}^* = C^+$. This means that C_n converges to $\tilde{C}^*$ in the cycle topology. In particular, for every sufficiently big n every irreducible component of C_n meets the interior $\mathsf{Int}(A)$ of A.

The latter property shows that the same argumentation can be used if we replace $\tilde{C}^*$ by any C_n with $n \gg 1$ and the lemma follows. □

Now we are ready to complete the

Proof of *Theorem 6.2.3*. It follows from the construction of the extension $\tilde{C}^*$ that the sequence (C_n, J_n) can be extended to a sequence $(\tilde{C}_n, \tilde{J}_n)$ such that $\tilde{J}_n$ is a sequence of ω-tamed almost complex structures in $\mathbb{CP}^2$ converging to $\tilde{J}^*$ and $\tilde{C}_n$ is a sequence of compact (*i.e.* closed) $\tilde{J}_n$-holomorphic curves converging to $\tilde{C}^*$. Moreover, we may additionally assume that the curves $\tilde{C}_n$ pass through the marked points $\boldsymbol{x}$ for all sufficiently big n.

Observe that all $\tilde{C}_n$ are symplectically isotopic. We denote by $\tilde{S}$ the closed oriented real surface parameterizing $\tilde{C}_n$. It can be obtained from the surface S parameterizing C_n by gluing in discs to fill out the holes in S.

Fix a sequence of homotopies $\{\tilde J_{n,t}\}_{t\in[0,1]}$ of almost complex structures with the following properties:

- all $\tilde J_{n,t}$ are C^ℓ-smooth and depend C^ℓ-smoothly on t;
- every initial structure $J_{n,0}$ is $\tilde J_n$;
- for some small $\varepsilon_0>0$ the structures $\tilde J_{n,t}$ are integrable in B for all $t\in[1-\varepsilon_0,1]$;
- as n goes to infinity, the structures $\tilde J_{n,t}$ converge to $\tilde J^*$ in the C^0-topology uniformly in $t\in[0,1]$, *i.e.*
$$\lim_{n\longrightarrow\infty}\sup_{t\in[0,1]}\|\tilde J_{n,t}-\tilde J^*\|_{C^0(\mathbb{CP}^2)}=0;$$
- the homotopy $\{\tilde J_{n,t}\}_{t\in[0,1]}$ is generic for every n.

Now let us try to deform continuously every $\tilde C_n$ inside a family $\tilde J_{n,t}$-holomorphic curves preserving the isotopy class. Since we want to control also the local isotopy class we must impose the condition that the curves in the family lie sufficiently close to $\tilde C^*$. Apriori, it can occur that such a curve does not exist for all $t\in[0,1]$. Nevertheless, we can find the maximal subinterval where such a family of curves exists. Moreover, we allow that under the deformation some nodal points appear. Let us formalize this observation.

Proposition 6.2.8. *Fix a sufficiently small $\varepsilon>0$. Then for every $n\gg1$ there exists a $t_n^+\in(0,1]$ which is maximal with respect to the following condition:*

For any $t<t_n^+$ there exists a $\tilde J_{n,t}$-holomorphic curve $\tilde C_{n,t}$ such that

- *$\tilde C_{n,t}$ passes through the fixed points $\boldsymbol{x}$ on $\mathbb{CP}^2$;*
- *the curve $\tilde C'_{n,t}$, obtained from $\tilde C_{n,t}$ by smoothing of all singular points contained in B, is symplectically isotopic to $\tilde C_n$;*
- $\mathsf{dist}_A(\tilde C_{n,t},C^*)<\varepsilon$.

Recall that for t sufficiently close to 1 the structures $\tilde J_{n,t}$ are integrable in B. So if $t_n^+=1$ for some n, then for some t close to 1 we obtain a *holomorphic* curve $C_{n,t}:=\tilde C_{n,t}\cap B$ whose smoothing is symplectically isotopic to the original curve C_n. In this case *Theorem 6.2.3* follows from *Lemma 6.2.4*. We claim that it is always the case for $n\gg1$.

To show this, let us analyze the possible reasons which could cause the strict inequality $t_n^+<1$ for a given $n\gg1$. Consider an increasing sequence of parameters t_ν approaching to t_n^+. Then there exists a sequence of $\tilde J_{n,t_\nu}$-holomorphic curves $\tilde C_{n,t_\nu}$ with the properties from *Proposition 6.2.8*. In particular, all $\tilde C_{n,t_\nu}$ are homologous to $\tilde C^*$. Taking a subsequence, we may assume that $\tilde C_{n,t_\nu}$ converges to a $\tilde J_{n,t_n^+}$-holomorphic curve $\tilde C_n^+$ in the cycle topology. Note $\mathsf{dist}_A(\tilde C_n^+,\tilde C^*)\leqslant\varepsilon$ by our construction. By *Lemma 6.2.7*, $\tilde C_n^+$ is sufficiently close to $\tilde C^*$ also with respect to the cycle topology. Consequently, near every nodal point of $\tilde C^*$ there is exactly one nodal point of $\tilde C_n^+$.

Observe that $\tilde C_n^+$ has no singular point $x_n^+\in\tilde C_n^+$ with the nodal number $\geqslant\delta^*$ at x_n^+. Indeed, otherwise we can repeat the argumentation from the proof of *Lemma 6.2.6* and show that $\tilde C_n^+$ must consist of rational components the number of which is the same as that for $\tilde C^*$. But the expected dimension of such curves in the space $\mathscr{M}'_{\tilde J_{n,t_n^+},\boldsymbol{x}}$ with a singular point of this type is negative and less then -1. So the existence of $x_n^+\in\tilde C_n^+$ with $\delta(x_n^+,\tilde C_n^+)\geqslant\delta^*$ contradicts the genericity of the path $\tilde J_{n,t}$. Thus all singularities of $\tilde C_n^+$ are simpler than those of $\tilde C^*$. By the induction assumption, the curve $\tilde C'_n$ obtained as the canonical smoothing of all singular points of $\tilde C_n^+$ contained in B is symplectically isotopic to $\tilde C_n$.

Let $u_n^+ : S_n^+ \to \mathbb{CP}^2$ be a *normal* parameterization of $\tilde{C}_n^+$ (see *Definition 5.2.3*). Consider the relative moduli space $\mathscr{M}_{h_n,\boldsymbol{x}}(S_n^+, \mathbb{CP}^2)$ of $h_n(t) = \tilde{J}_{n,t}$-holomorphic curves which are parameterized by S_n^+, are in the homology class $[\tilde{C}^*]$, and pass through the fixed points $\boldsymbol{x}$. This space is non-empty since it contains $(\tilde{C}_n^+, t_n^+)$. *Theorem 4.5.3* provides that for some interval $t \in [t_n^+, t_n^{++}]$ with $t_n^{++} > t_n^+$ we can construct a path of $\tilde{J}_{n,t}$-holomorphic curves $\tilde{C}_{n,t}$ which lies in $\mathscr{M}_{h_n,\boldsymbol{x}}(S_n^+, \mathbb{CP}^2)$ and starts at $\tilde{C}_n^+$. Then the curves obtained from such $\tilde{C}_{n,t}$ by smoothing of all singular points contained in B will be symplectically isotopic to $\tilde{C}_n$. Moreover, if we would additionally have the strict inequality $\mathsf{dist}_A(\tilde{C}_n^+, \tilde{C}^*) < \varepsilon$, then $\mathsf{dist}_A(\tilde{C}_{n,t}, \tilde{C}^*) < \varepsilon$ for some $t \in]t_n^+, t_n^{++}[$, and this would contradict the maximality of t_n^+.

Thus we may assume that $\mathsf{dist}_A(\tilde{C}_n^+, \tilde{C}^*) = \varepsilon$ for every n. Then for every $n \gg 1$ we can fix $t_n^- \in [0, t_n^+]$ and a $\tilde{J}_{n,t_n^-}$-holomorphic curve $\tilde{C}_{n,t_n^-}$ which has properties from *Proposition 6.2.8* and satisfies the additional condition

$$\frac{\varepsilon}{2} \leqslant \mathsf{dist}_A(\tilde{C}_{n,t_n^-}, \tilde{C}^*) \leqslant \varepsilon.$$

Taking a subsequence, we may assume that $\tilde{C}_{n,t_n^-}$ converges to a $\tilde{J}$-holomorphic curve $\tilde{C}^+$ in the cycle topology. Then

$$\frac{\varepsilon}{2} \leqslant \mathsf{dist}_A(\tilde{C}^+, \tilde{C}^*) \leqslant \varepsilon.$$

By *Lemmas 6.2.6* and *6.2.7*, $\tilde{C}^+$ must have simpler singularities than $\tilde{C}^*$ provided the constant ε was chosen small enough. By the induction assumption, the curve $\tilde{C}'$ obtained by canonical smoothing of all singular points of $\tilde{C}^+$ lying in B is symplectically isotopic to every $\tilde{C}_n$, as also to every $\tilde{C}_{n,t_n^-}$. On the other hand, $\tilde{J}^*$ coincide in B with the standard structure J_{st}. Thus $C' := \tilde{C}' \cap B$ is a canonical smoothing of C^* by *Lemma 6.2.4*.

6.3. **Global symplectic isotopy in $\mathbb{CP}^2$.** In this paragraph give

Proof of *Theorem 6.1.2*. We proceed by making appropriate modifications of the argumentation used in the proof of *Theorem 6.2.3*. Let Σ be an imbedded surface in $\mathbb{CP}^2$ of degree $d \leqslant 6$, such that $\omega_{\mathsf{st}|\Sigma}$ is non-degenerate. By *Proposition 6.1.3*, to prove the theorem it is sufficient to show that Σ is symplectically isotopic to a non-singular algebraic curve of degree d.

Find an ω_{st}-tame almost complex structure J_0 making Σ a J_0-holomorphic curve, denoted by C_0. Fix $3d-1$ distinct points $\boldsymbol{x} = (x_1, \ldots, x_{3d-1})$ on C_0. Perturbing C_0 and the points, we may assume that $x_1, \ldots, x_{3d-1}$ are in generic position with respect to the standard structure J_{st} in the following sense. For any positive degree $d' \leqslant d$ and any closed oriented surface S, not necessary connected, the moduli space $\mathscr{M}_{J_{\mathsf{st}},\boldsymbol{x}}(S, \mathbb{CP}^2; d')$ of J_{st}-holomorphic (and hence *algebraic*) curves of degree d' with normalization S passing through $\boldsymbol{x}$ is a (possibly empty) complex space of the expected dimension.

Fix a generic path $h(t)$ of ω_{st}-tame almost complex structures $J_t := h(t)$ connecting J_0 with $J_{\mathsf{st}} = J_1$. Without loss of generality we may assume that all J_t are C^ℓ-smooth and depend C^ℓ-smoothly on t.

Proposition 6.3.1. *There exists a $t^+ \in (0,1]$ which is maximal with respect to the following condition:*

For any $t < t^+$ there exists a J_t-holomorphic curve C_t such that

i) C_t passes through the fixed points $\boldsymbol{x}$ on $\mathbb{CP}^2$;

ii) C_t is non-multiple, but not necessarily irreducible;

iii) the curve C_t', obtained from C_t by smoothing of all singular points, is symplectically isotopic to C_0.

To prove the theorem, we must show that $t^+ = 1$ and that there exist a J_{st}-holomorphic curve C_1 with the properties *i)–ii)* in the proposition.

Let t_n be an increasing sequence converging to t^+. Fix J_{t_n}-holomorphic curves C_n with these properties. Property *ii)* implies that the C_n have the same degree d. Going to a subsequence we may assume that they converge to a J_{t^+}-holomorphic curve C^+ in the cycle topology.

We claim that C^+ contains no multiple components. To show this, it is sufficient to consider only the case when C^+ has only two components C_1^+ and C_2^+ with multiplicities $m_1 = 1$ and $m_2 = 2$ respectively. Let d_i be the degree of C_i^+, so that $d_1 + 2d_2 = d$. Then the geometric genus g_i of every C_i is at most $g_i \leqslant \frac{(d_i-1)(d_i-2)}{2}$. By the genericity of the path $h(t) = J_t$, each C_i can contain at most $k_i \leqslant 3d_i - 1 + g_i \leqslant \frac{d_i(d_i+3)}{2}$ of the fixed points $\boldsymbol{x}$, see *Paragraph 2.4*. Thus C^+ can contain at most $\leqslant \frac{d_1(d_1+3)+d_2(d_2+3)}{2}$ points. It is easy to show that for $d \leqslant 6$ this number is strictly less then $3d-1$. For example, in the worst case with $d = 6$, $d_1 = 4$, and $d_2 = 1$ we would have on C^+ at most $\frac{4\cdot(4+3)}{2} + \frac{1\cdot(1+3)}{2} = 14 + 2 = 16$ the marked points $\boldsymbol{x}$ instead of the necessary $3\cdot 6 - 1 = 17$. Observe, that this argument remains valid also in the case $t^+ = 1$ and $J_1 = J_{\mathsf{st}}$.

Now the results of *Paragraph 6.2* show that the curve C' obtained from C^+ by the canonical smoothing of all singular points is symplectically isotopic to C_0. This implies the theorem in the case $t^+ = 1$. Indeed, in this case C^+ is the zero divisor of a homogeneous polynomial F^+ of degree d. Making a generic perturbation of coefficients of F^+ we obtain a polynomial F' whose zero divisor is an algebraic (and hence J_{st}-holomorphic) curve C' which is symplectically isotopic to C_0.

In the case $t^+ < 1$ we must show that for some $t^{++} > t^+$ there exists a $J_{t^{++}}$-holomorphic curve C^{++} with the properties given in *Proposition 6.3.1*. To do this we fix a normal parameterization $u^+ : S^+ \to C^+ \subset \mathbb{CP}^2$ (see *Definition 5.2.3*) and consider the relative moduli space $\mathscr{M}_{h,\boldsymbol{x}}(S^+, \mathbb{CP}^2, d)$ of $J_t = h(t)$-holomorphic curves which are parameterized by S^+, pass through $\boldsymbol{x}$ and have the degree d. This space is non-empty because it contains C^+. The results of *Paragraph 4.5* imply that for some interval $t \in [t_n^+, t_n^{++}]$ with $t_n^{++} > t_n^+$ we can construct a path of J_t-holomorphic curves C_t^+ which lies in $\mathscr{M}_{h,\boldsymbol{x}}(S^+, \mathbb{CP}^2, d)$ and starts at C^+. By *Paragraph 6.2*, the C_t^+ have the the properties *i)–iii)* from *Proposition 6.3.1*. This contradicts the maximality of t^+ and implies the statement of *Theorem 6.1.2*. □

Remark. In fact, the real homotopy C_t from $C_0 = \Sigma$ to an algebraic curve C_1 has the property described in *Paragraph 6.1*. Namely, after fixing a generic homotopy $h(t) = J_t$, one tries to construct *any* path C_t of *imbedded* J_t-holomorphic curves C_t. Such a path exists for some interval $t \in [0, t')$. The saddle point property proven in *Paragraph 4.5* removes the main difficulty in the construction of C_t: the presence of local maxima in the corresponding moduli space $\mathscr{M}$. This means that at end of this interval, when $t \longrightarrow t'$, the curves C_t go to infinity in $\mathscr{M}$. By Gromov compactness, going along some sequence $t'_n \longrightarrow t'$, we approach a $J_{t'}$-holomorphic curve C' lying on some infinity stratum of $\overline{\mathscr{M}}$ parameterized by a new moduli space $\mathscr{M}'$. As we have shown in the proof, one can avoid the strata $\mathscr{M}'$ corresponding to curves with multiple components. Now we continue to deform C' as a path C'_t along $\mathscr{M}'$, having in mind that the canonical (in fact, any) smoothing of singular points of C'_t gives curves symplectically isotopic to C_0. The new path C'_t continues until we come to the next infinity stratum $\mathscr{M}''$ of $\overline{\mathscr{M}}$, and so on.

We finish the paper with a remark on *Conjecture 2* about the local symplectic isotopy problem for nodal curves. The proof of this result would follow from the corresponding

result for *holomorphic* curves, which is essentially a local version of the Severi problem, see *Proposition 6.1.3*. Indeed, the proof of *Theorem 6.2.3* could be applied after appropriate modification.

Conjecture 3. *(Local Severi-Harris problem). Let C^* be a holomorphic curve in the ball $B \subset \mathbb{C}^2$ with a unique isolated singular point at $0 \in B$ and without multiple components.*

Then any nodal holomorphic curve $C \subset B$ sufficiently close to C^* with respect to the cycle topology is holomorphically isotopic to a holomorphic curve $C^\dagger$ obtained from a maximal nodal deformation C_{nod} of C^* by smoothing some number of nodes on C_{nod}.

In view of the main results of this paper, the validity of *Conjecture 3*, and hence of *Conjecture 2*, seems quite plausible.

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